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GEOMETRY FOR TODAY

BASED ON

A JUNIOR GEOMETRY

BY

A. W. SIDDONS AND R. T. HUGHES

BY

ALEXANDER J. COOK
Associate Professor of Mathematics
University of Alberta

Authorized by the Department of Education for Alberta

TORONTO: THE MACMILLAN COMPANY OF
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1945

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PREFACE

THIS book is based upon the well-known English text: *A Junior Geometry*, A. W. Siddons and R. T. Hughes (Cambridge University Press, 1930). Also, some use was made of *A Practical and Theoretical Geometry* (Cambridge, 1923) by the same authors and of *A New Geometry*, A. W. Siddons and K. S. Snell (Cambridge, 1938).

Important features of the original text have been preserved. Certain other new features appear in the present text. Some of these are: (1) the role of theory has been illustrated in simple terms; (2) the right triangle has been given more prominence; (3) the treatment of similar figures has been re-designed; (4) the definition of a tangent to any curve has been made the starting point for tangent properties of circles.

The book pre-supposes some knowledge of geometry, but the first six chapters are probably, in the main, familiar to the student. They should be covered rapidly, for purposes of reference and review.

The first ten chapters deal with the theory of the straight line. Happily, this simple theory is adequate for a beginner's approach to the logical aspects of everyday thinking and some attention is paid to these matters in the text.

Hints for the proofs of theorems and constructions which belong to this early theory are given in Chapter X. It is hoped that these hints will be consulted only when the student is in difficulty; a proof is more respected when it is approached through personal trial, failure and success. As the student begins to gain some appreciation of how his proofs rest fundamentally on the use of undefined words and unproved propositions, his earlier confidence may be somewhat shaken. But proofs do make the propositions hang together; there is system and order, even if the system is not rigid and the order is not irrevocable. Indeed, the student should begin to understand that theories and systems are not certainties or absolutes, but tools in human hands which may be used for the furtherance or hindrance of human welfare.

Chapters XI-XIII on the circle have been treated somewhat differently from those on the straight line. Here the ideas are less familiar and the student is encouraged to explore and to organize. Consequently he should work through each property in detail, devise his own statements of findings and construct his own proofs. His work may be checked against the formal statements of the theorems which are given at the end of each chapter. The exercises are plentiful and are numerical for the most part.

A manual for the use of teachers in the High Schools of Alberta has been prepared and contains a list of reference books for the teacher. We should like to draw attention here, however, to (1) the two reports on the teaching of geometry of the British Mathematical Association (1923 and 1938, Bell and Sons, London); (2) the 1930 and 1938 yearbooks of the U.S. National Council of Teachers of Mathematics, on teaching and on the nature of proof (Columbia University Press); (3) a valuable, if somewhat more technical, book on logic and geometry by Lazar (Banta Publishing Co., Menasha, Wis.).

The illustrations are the work of Miss Joan Fraser, a grade IX student in the Normal Practice School at Edmonton, and were drawn under the direction of M. W. Macdonald, the Principal of the school.

The text figures were prepared by the publishers.

The section on engineering constructions in Chapter V was prepared in consultation with Professor H. R. Webb, of the Department of Civil Engineering of the University of Alberta.

Revision exercises (F) were contributed by S. C. Clarke, Principal of the High School at Two Hills, Alberta.

The text was written at the direction of the Mathematics Committee of the Department of Education and the committee desires me to express its thanks to R. S. Sheppard, Assistant Superintendent of Schools, Edmonton, and to J. W. Churchill, of the Calgary teaching staff, for their earlier work on the revision of the English text. It should be pointed out, however, that apart from one or two exercises on circular arches no use of these early notes was made by the author of the present text.

The writer is greatly indebted to the members of the Mathematics Committee and to the Revision Committee appointed to review the manuscript, for their many helpful criticisms.

Miss V. McMahan, of the High School Correspondence Branch, read the text in proof and her corrections and suggestions are gratefully acknowledged.

My colleague, Professor E. W. Sheldon, has greatly assisted me with detailed criticisms based on his knowledge of the implications and limitations of logical thinking.

My special thanks are due to Inspector H. E. Balfour of the Department of Education for his many services of counsel and criticism so generously given throughout the preparation of the text.

Finally, it is a pleasure to acknowledge the courteous and efficient co-operation of the editorial and manufacturing departments of The Macmillan Company of Canada Limited.

A. J. Cook.

Edmonton,
June, 1940.

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CHAPTER I

ANGLES, PARALLELS AND PERPENDICULARS

1. Angles.

You know already a good deal about angles. Try the quiz which follows.

QUIZ*

Is it true or is it false that

1. Complementary angles are two acute angles whose sum is 90°?

2. A straight line a which crosses another straight line b at 30° will also cross a straight line parallel to b at 30°?

3. Two angles which have the same vertex and a common side are adjacent angles?

4. The sum of two adjacent angles is two right angles?

5. The supplement of 50° is obtained by adding 90° to 50°?

6. The supplement of any angle is an obtuse angle?

7. A straight angle is the angle between two perpendicular straight lines?

8. No matter how a straight line crosses two parallel straight lines, the sum of the interior angles on either side of the line is always 180°?

9. When a straight line crosses each of two other straight lines at an angle of 40°, these two lines must be parallel?

10. Vertically opposite angles are the angles at the opposite vertices of a parallelogram?

11. An aeroplane propeller turning at a speed of 1800 r.p.m. turns through 120 right angles every second?

12. Each point on the perpendicular bisector of the line segment RS is at the same fixed distance from R and S?

---

*If you don't know the answer, say so. Answers to this and subsequent quizzes are stated as false if they are not always true.

13. The supplements of a pair of supplementary angles are equal?

14. When a bridge crosses a straight river 150 ft. above the river, the centre line of the bridge floor and the centre line of the river surface are parallel straight lines?

2. A surveying problem.

Runways were laid out recently at the local airport. The layout is shown in fig. 1. The layout was made with stakes, picket, chain and transit as follows:

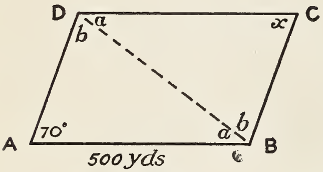
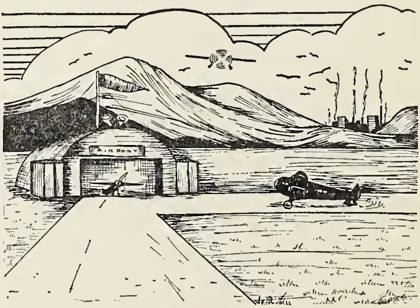


Fig. 1.1

(i) Measure AB = 500 yd. due east from A to locate B.
(ii) Turn off ∠ BAD = 70° from AB to AD.
(iii) Measure AD = 200 yd. to locate D.
(iv) Turn off ∠ ADC = 110°.
(v) Measure DC = 500 yd. to locate C.
(vi) Turn off ∠ DCB = 70° to sight B (Check).

Bill, a friend of the surveyor, helped him with the layout. Bill wondered why ∠ C = 70° = ∠ A. The surveyor thought a moment, drew a letter *N* on the ground and asked Bill what he knew about the angles in *N*. He then drew fig. 1 on paper. Bill saw at once the two pairs of equal angles shown, but still could not prove ∠ A = ∠ C. Then his friend asked him, "What



do you know about the angles of a triangle?" Bill knew. The surveyor then wrote down the following equations:—

$$\begin{aligned} a+b+x &= 180°, \\ a+b+70° &= 180°, \\ \therefore x &= 70°. \end{aligned}$$

Bill then said, "Now I see that $\angle$ A = $\angle$ C, but how did you know to turn off $\angle$ D = 110°?" His friend smiled and said, "One of the equations I wrote just now also proves that $\angle$ D = 110°."

In the above proof Bill had to know two things:—

(1) When a straight line cuts two parallel straight lines, the alternate angles are equal.

(2) The sum of the angles of a triangle is two right angles.

He was then able to prove from (1) and (2) that:—

(3) Opposite angles in a parallelogram are equal.

(4) When a straight line cuts two parallel straight lines, the interior angles on the same side of a transversal are supplementary.

His friend the surveyor used (5) the *converse* of (4) in the layout.

(5) If a straight line cuts two straight lines and the interior angles on the same side of the transversal are supplementary then the pair of straight lines are parallel.

EXERCISES. A Surveying Problem

1. Had Bill known (4) to be true he could have proved (3) directly. Why?

2. When the layout was complete with stakes at **A**, **B**, **C** and **D**, the surveyor asked Bill if he thought **AB** was parallel to **CD**. Bill said that he wasn't so sure now for the distances were great and they didn't look parallel. The surveyor suggested that they chain the distances again. This done they found that **AB** = **CD** = 500 yd. and **AD** = **BC** = 200 yd. The surveyor said: "Fine, then we know that the opposite sides are parallel." Bill again wanted to know why and his friend said: "You can prove it, if you know that two triangles are equal in every respect whenever corresponding sides of the two triangles, each to each, are equal." He continued, "You will also want to remember letter *N*." Can you show that the surveyor was right?

3. Angle theory.

At the end of the chapter you will find a summary of the various facts about angles, parallels and perpendiculars. You are probably familiar with these. You can prove some in terms of others. Thus you can prove Theorem 3 if you are satisfied that Theorem 1 is true. Also you can prove Theorem 4 (iii) (assumed by the surveyor in § 2) if you assume Theorem 4 (i). You will notice that a new assumption, often called Playfair's Axiom, is needed before Theorem 5 can be proved. The proof of Theorem 5 (i) is shown below:—

THEOREM 5 (i) (CONVERSE OF THEOREM 4 (i)). **If a straight line cuts two parallel straight lines, corresponding angles are equal.**

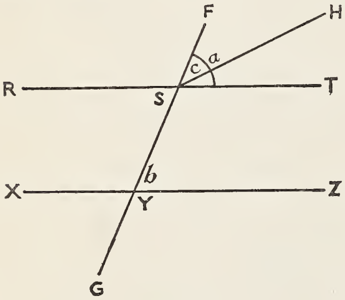


Fig. 1.2

Given FG is a line cutting the parallel straight lines RT, XZ at S, Y.

To prove that $\angle a =$ cor. $\angle b$.

Construction. Either (i) $\angle a = \angle b$ or (ii) $\angle a \neq \angle b$.* Suppose (ii). Then draw SH to make $\angle c$ ($\angle$FSH) $= \angle b$ (Const. 1).

Proof. SH' is parallel to XZ (Th. 4 (i)).

Hence through S there are two straight lines RT and SH parallel to XZ. This contradicts the assumption noted above, so that (ii) is false and (i) is true.

$\therefore$ Angle $a =$ cor. Angle b.

---

*$\neq$ means "not equal to".

The method of this proof consists of eliminating one of two possibilities. It is often called the method of elimination. This method was applied to the study of geometry by the followers of the Greek philosopher Plato (380 B.C.).

EXERCISES. Angle theory

1. Prove (a) Theorem 4 (iii) (b) Theorem 5 (ii) and (iii) knowing Theorem 5 (i).

2. Assume Theorem 5 (i). What else do you need to know in order to prove Theorem 4 (i)?

3. John said, "I could prove Theorem 5 (i) if I could assume that a line which crosses RT at right angles crossed XZ also at right angles." "Of course," he added, "I know that the angles of a triangle add up to 180°." Do you see how John proved the theorem? Is his first assumption a special case of Theorem 5 (i)?

4. A dilemma indicates just two possibilities; either *A* or else *B*. Some of the following either-or statements are incorrect. Find these and indicate the other possibilities present; not either *A* or *B*, but one of *A*, *B*, *C*, ...

(a) Either I am 6 ft. tall or I am not 6 ft. tall.
(b) In the plane either a straight line p is || to a straight line q or else p is not parallel to q.
(c) The roof is either red or green.
(d) Either I take it or leave it.
(e) That youngster will either be a convict or a millionaire.
(f) Given two numbers x and y. Either $x > y$ or else $x < y$.
(g) The edges of a box are either || to one another or they intersect one another.

5. "A friendly warning to screen buyers! Bronze screens keep them (flies and mosquitos) out." Criticize the following argument.

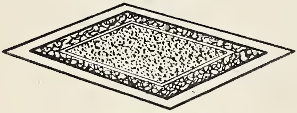
Either screens keep them out or screens do not keep them out.
Either screens are bronze or screens are not bronze.

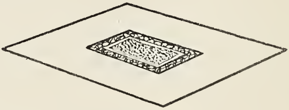
Bronze screens keep them out.
∴ Non-bronze screens do not keep them out.

6. "Drive this smart, new, lowest price car." Criticize the following argument.

A new car is sold at the lowest price or else at a higher price.
This car sold at the lowest price.
∴ Every other make of car sold at a higher price.

7. A carpet manufacturer's advertisement showed carpeted floors as in the figures. (a) was marked right because "it really fits". (b) was marked wrong because "it looks skimpy". Do you agree that if (a) is right then (b) is wrong? Do you agree that (a) is right? That (b) is wrong?


(a)


(b)

4. Exercises. Angle properties.

In the following exercises you are to assume as known any of the facts in the summary at the end of the chapter. No other assumptions are needed. Thus, you need not assume knowledge of triangles. You may, however, if you wish, use the angle sum property of a triangle. Careful freehand figures should be made and your reasons should be clearly stated.

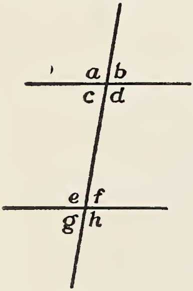


Fig. 1.3

1. In fig. 3 prove that:

(i) if $\angle c = \angle f$ then $\angle d = \angle e$;
(ii) if $\angle d + \angle f = 2$ *rt.* $\angle$s then $\angle b = \angle f$;
(iii) if $\angle g = \angle c$ then $\angle d = \angle h$;
(iv) if $\angle h = \angle a$ then $\angle e = \angle d$;
(v) if $\angle a = \angle e$ then $\angle b = \angle g$.

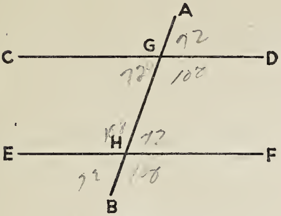


Fig. 1.4

2. Copy fig. 4 freehand. If CD, EF are parallel and ∠ AGD $=72°$, find all the angles in the figure, giving your reasons; make a table of values.

3. Repeat ex. 2 with ∠ FHB $=120°$, CD, EF being parallel.

4. In fig. 5 there are two pairs of parallel lines; prove that the following pairs of angles are equal:—(i) b, l, (ii) f, k, (iii) m, s, (iv) f, h, (v), r, l, (vi) s, h, (vii) s, q, (viii) s, k, (ix) s, a, (x) g, l.

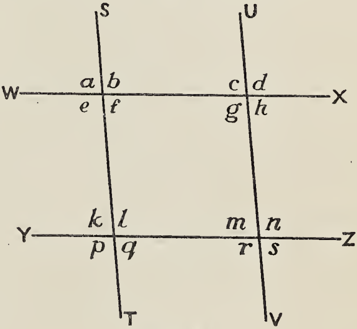


Fig. 1.5

(State your reasons carefully.

e.g., WX, YZ are || and ST cuts them, ∴ ∠ q = cor. ∠ f.)

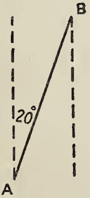


Fig. 1.6

5. The bearing of B from A is N. 20° E. What is the bearing of A from B? See fig. 6.

6. A submarine is observed by a destroyer to bear S. 30° E. What is the bearing of the destroyer from the submarine?

7. In fig. 7 **FG** is parallel to **BC**; find all the angles in the figure, stating your reasons carefully.

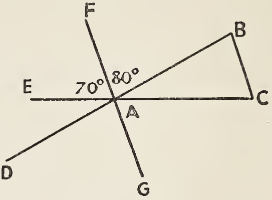


Fig. 1.7

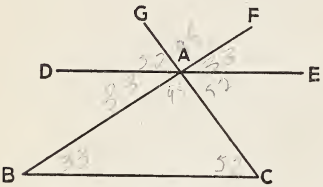


Fig. 1.8

8. If, in fig. 8, **DAE** is parallel to **BC**, ∠ **B** = 33°, ∠ **C** = 52°. Find all the angles in the figure, giving your reasons in each case.

9. In fig. 9 **PQ** and **RS** are parallel. If ∠ a = 42° and ∠ b = 47°, find ∠ x. (Through **O** draw a parallel to **PQ** and **RS**.)

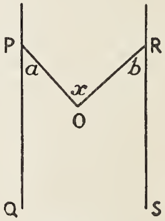


Fig. 1.9

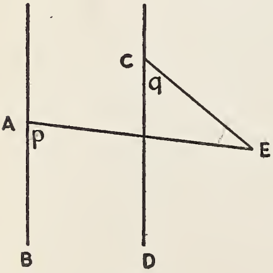


Fig. 1.10

10. In fig. 10 **AB** is || to **CD**. Find the angle at **E** in terms of p and q.

11. In fig. 11 **BE** is || to **AF**. **BC** bisects the angle **ABE**. ∠ **A** = 60°. Find all the other angles.

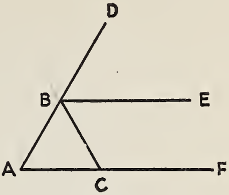


Fig. 1.11

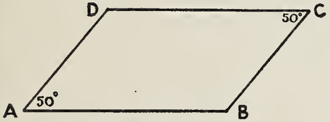


Fig. 1.12

12. In fig. 12 **AD** is || to **BC**. Show that **DC** is || to **AB**.

13. In fig. 13 **AD** is || to **BC** and **CD** to **BA**. Find all the other angles, giving your reasons.

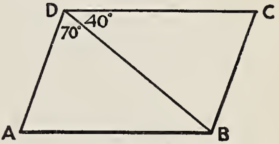


Fig. 1.13

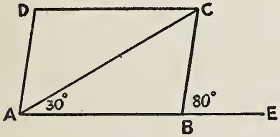


Fig. 1.14

14. Repeat ex. 13 for fig. 14.

15. In fig. 15, **AC** is || to **BD** and **AB** to **CD**. If ∠ **B** = 72° and ∠ **BAD** = 78°, find the other angles.

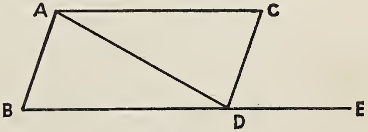


Fig. 1.15

5. Point, line, plane in space.

Thus far we have been thinking about points and lines in a plane. The world of our immediate surroundings, however, is crowded with figures which show lines and points in different planes. Thus, house walls are "planes", the eaves are "straight lines", and the gable corners are "points".

The facts about lines, points and planes are needed in surprising ways. The earth is for nearly all scientific purposes a sphere. The theory of navigation now used by the air pilot as well as by the seaman is based on the geometry of the sphere. This geometry uses planes and lines. For instance, a great circle on a sphere is the section of a sphere by a plane through the centre of the sphere. A spherical triangle is the figure formed on the sphere by three planes through the centre intersecting by pairs in three straight lines (diameter of the sphere).

EXERCISES. Point, Line, Plane in Space

Use familiar objects such as pencils and cardboard to visualize lines and planes in space. Draw plentiful freehand figures. A plane is usually shown as a parallelogram as in fig. 16. Show hidden parts of figures dotted.

1. Complete the following, with illustrations.

   (a) (Sample). A straight line may or may not intersect a plane. When it intersects the plane, the intersection is a *point*. If it does not intersect the plane the line is said to be *parallel* to the plane. (See fig. 16).
   Illustrations:— (i) A tree growing on a hillside;
   (ii) The water main under a street.

   (b) Two planes may or may not intersect. When the planes intersect, the intersection is a........... If the planes do not intersect they are said to be.......

   (c) A straight line is located definitely in space by...... points, or by ....planes.

(d) A plane is located definitely by two..........straight lines or by two..........straight lines or by ...... points, if these points do not.........

(e) Two straight lines may or may not intersect. If they do not intersect they may be........or.........

(f) Given two parallel straight lines. A plane through one of them is either........the other or.......... the other.

(g) Given a plane and a straight line parallel to it. A second plane through the line rotates about the line. The intersections of the rotating plane with the given plane are...........

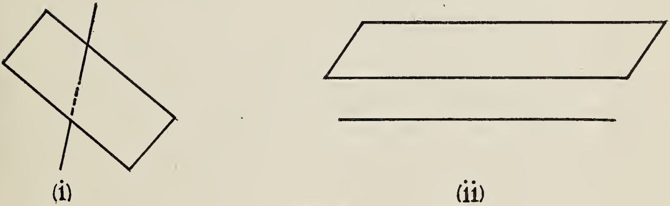

(i) (ii)

Fig. 1.16

2. Find examples about you of:

(a) two perpendicular straight lines;
(b) three perpendicular straight lines;
(c) four straight lines perpendicular to one straight line;
(d) a straight line perpendicular to a plane;
(e) two straight lines each perpendicular to a plane;
(f) a straight line perpendicular to each of two planes;
(g) two perpendicular planes;
(h) three planes, each perpendicular to the other;
(i) a plane perpendicular to three planes.

I. Angles, Parallels, Perpendiculars

A. ANGLES

Terms:

Acute, right, obtuse, straight angles.
Complementary and supplementary pairs of angles.
Adjacent angles; vertically opposite angles.

B. TWO INTERSECTING STRAIGHT LINES

‡THEOREM 1. **The sum of any pair of adjacent angles formed by two intersecting lines is equal to two right angles.**

COROLLARY. **If any number of straight lines intersect in a point, the sum of all the consecutive adjacent angles is equal to four right angles.**

‡THEOREM 2. (CONVERSE OF THEOREM 1). **If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.**

†THEOREM 3. **If two straight lines intersect, the vertically opposite angles are equal.**

C. TWO STRAIGHT LINES CUT BY A THIRD STRAIGHT LINE

Terms:

Parallel straight lines; transversal; alternate, interior, exterior, corresponding angles.

THEOREM 4. **Two straight lines are cut by a transversal.**

‡(i) **If a pair of corresponding angles are equal the two lines are parallel.**

(ii) **If a pair of alternate angles are equal the two lines are parallel.**

---

‡ Propositions marked ‡ are treated in this book as assumptions. Although they are labelled "theorems", they are to be taken for granted.

† Propositions marked † may be treated as assumptions. Any one of these, however, is capable of proof in terms of propositions which precede it.

(iii) **If a pair of interior angles on the same side of the transversal are together equal to two right angles, the two lines are parallel.**

Corollary. **If each of two straight lines is perpendicular to a third straight line, the two lines are parallel to one another.**

Assumption.

Through a given point only one straight line can be drawn parallel to a given straight line.*

†Theorem 5. (Converse of Theorem 4.) **If two parallel straight lines be cut by a transversal,**

(i) **corresponding angles are equal;**
(ii) **alternate angles are equal;**
(iii) **the interior angles on the same side of the transversal are together equal to two right angles.**

Corollary. **Straight lines which are parallel to the same straight line are parallel to each other.**

The following constructions are added for completeness. They will be discussed in chap. V.

D. CONSTRUCTIONS

1. *To make an angle at* O *equal to an angle at* A.

2. *To bisect an angle.*

3. *To draw the perpendicular bisector of a straight line segment.*

4. *To draw a perpendicular through a point to a straight line.* (*The point is any point in the plane and hence may or may not be on the line*).

5. *To draw a straight line through a point parallel to a straight line* (*see the assumption under C above*).

---

*To prove Theorem 5 it is necessary to make this assumption or some other which although looking different would really say the same thing.

CHAPTER II

THE RIGHT TRIANGLE

1. Facts concerning the right triangle.

Triangles with one right angle are of common occurrence, and the facts concerning them are likely well known to you. Such a triangle RST has three sides r, s, t and three angles R, S, T with $T=90°$ as in the figure. Since $R+S+T=180°$, then $R+S=90°$ and $R=90°-S$ or $S=90°-R$.*

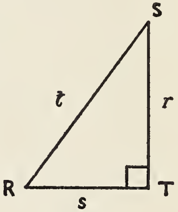


Fig. 2.1

A right triangle may be constructed with the following specifications:

(i) Any two sides given;
(ii) One side, one acute angle given.

Distance and angle problems involving right triangles may be solved by scale drawing to these specifications, as in the succeeding paragraphs.

Any two triangles drawn to the same specifications are of course identical figures. Such figures are said to be *congruent* figures.

EXERCISES. Facts concerning the right triangle

1. Give instances of the common use of the right triangle.

2. Draw, with careful but freehand figures, the following triangles according to the following specifications.

(a) ΔRST, $T=90°$;

(i) $R=45°$, $s=3$ in.; (iii) $r=2$ in. $s=1.5$ in.;
(ii) $R=30°$, $r=0.75$ in.; (iv) $s=2.5$ in. $t=4$ in.

---

*We assume here that the sum of the three angles of any triangle is 180°.

(b) Δ**XYZ**, **Z**$=90°$.

(i) $x=3$ in. $z=5$ in.; (ii) **Y**$=20°$, $y=1$ in.

3. In each instance draw three triangles **RST**, **T**$=90°$ with the following specifications:

(a) $r=s$;
(b) **R**$=30°$;
(c) $t=2s$;

2. Scale drawing and measurement.

In the construction of figures to scale according to specifications the following points should be noted:—

(i) State the problem.

(ii) Make a freehand sketch of the required figure. This should be done on the same sheet as the finished drawing, and should show the *data* (given things) clearly.

(iii) Make an accurate drawing. Choose as large a scale as possible; but not so large that any line in the figure is more than 5 in. long. Scales which involve only division or multiplication by 10 or 100 are the simplest. The next best are those that involve division or multiplication by 2, 20, 5 or 50. Avoid division or multiplication by 3, 6, 7, 8 etc. The scale used should be stated, viz., 1 in. to 1 mi.

(iv) The answer should be given in words; if a distance, it should be given in the original unit.

You should also know the limits of accuracy of your instruments.

EXERCISES. Scale drawing and measurement

HEIGHT, DISTANCE, BEARING

In bearing questions a N. *and* S. *line should first be drawn through the point from which the bearing is to be measured. The* **bearing** *of a place B from a place A is the point of the compass towards which an observer at A would be facing if he were looking in the direction of B.*

By "N. 10° W." *or* "10° W. *of* N." *is meant the direction in which you would be looking if you first faced due north and then turned through an angle of* 10° *towards the west.*

1. Find, by drawing, the length of a man's shadow if the man is 6 feet high and the altitude of the sun is 37°.

2. A man 6 feet high throws a shadow 8 feet long. By drawing a figure to scale, find the height of a man whose shadow is 7 feet long.

3. Fig. 2 is a sketch of a road which consists of three straight parts. The length and direction of each part of the road were measured and found to be as follows:— **AB** 6.4 miles, 23° east of north; **BC** 2.8 miles 40° east of north; **CD** 5.7 miles, 74° east of north.

Find the distance in miles direct from **A** to **D**, and the bearing of **D** from **A**.

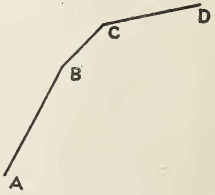


Fig. 2.2

4. **X** is 17.5 miles N.W. of **Y**; **Y** is 23 miles N.E. of **Z**. What is the distance and bearing of **X** from **Z**?

5. A radio mast is braced by two equal stays on opposite sides of the mast. The stays are each 125 ft. long and make an angle of 65° with the ground. Find: (i) the height of the mast; (ii) the distance between the ends of the stays.

The following exercises may be drawn to scale on co-ordinate paper if desired.

6. A ladder 10 ft. long rests on the ground and leans over a wall that is 6 ft. high. If the foot of the ladder is 2 ft. from the wall, how high is the top of the ladder above the ground?

7. A rod AB 6 ft. long hangs vertically down from A. If it is now turned about A until it makes an angle of 40° with the vertical, through what vertical height has B risen?

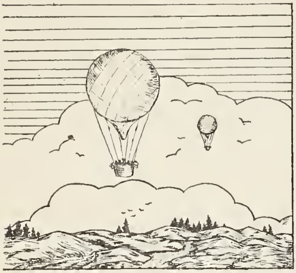
8. The end view of a garage shows two side walls each 7 ft. high and two equal sloping sides of the roof. If the width of the garage is 15 ft. and the height of the highest point of the roof $11\frac{1}{2}$ ft., what is the angle of slope of the roof and the length of the slope?

9. A rectangular drawing board 18″ by 14″ lies on a table with one corner on the edge of the table and the long side making an angle of 30° with the edge. How far are the other corners from the edge?

10. A horizontal trap-door measures 2 ft. 6 in. by 1 ft. 9 in., the hinges being along the shorter edge. What is the height of the edge opposite the hinges above the horizontal when the door is opened through an angle of 50°?

11. The city of Honolulu lies on the island of Oahu. Upolu point is the northernmost tip of the island of Hawaii. On the map this point is east 4.05 inches and south 2.27 inches from Honolulu. What is the bearing of Upolu from Honolulu?

12. An aeroplane flying at an altitude of 6000 ft. is simultaneously revealed by two searchlights which are at the moment due south of it. The nearer searchlight is inclined at an angle of 42° with the horizontal. At what angle is the farther inclined if the distance between the searchlights is 1000 yd.?



13. An observer in a balloon, one mile high, observes the angle of depression of a prominent building to be 35°. After ascending vertically for 20 min., he observes the angle of depression to be now 55.5°. Find the rate of ascent in miles per hour.

Three Dimensions

14. A spire is 40 ft. high, with a square base 20 ft. on side. It is desired to sheet the tower with tin. How many square feet of tin should be used (the area of any triangle is half the product of base and altitude)?

15. The sun is S. 45° W. The altitude of the sun is 40°. Find the width of the shadow of a 6-foot garden fence, which runs due east.

16. A steel hydro-electric tower is 75 ft. high. On the office map it is located as shown in fig. 3, where O is a "bench mark" on the ground. Find the distance and angle of the top of the tower from O.

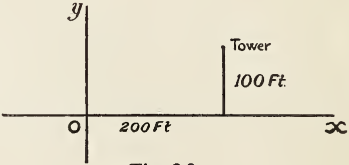


Fig. 2.3

3. The Pythagorean Theorem.

You will recall that a definite relation exists among the sides of a right triangle:

The square on the hypotenuse (the side opposite the right angle) is equal to the sum of the squares on the sides containing the right angle.

Thus in Δ RST with T = 90°, the sides r, s, t satisfy the equation $t^2 = r^2 + s^2$.

EXERCISES. The Pythagorean Theorem

1. What is the square on the hypotenuse of a right-angled triangle if the sides containing the right angle are 5 cm. and 12 cm.? Hence, calculate the length of the hypotenuse.

2. Find the hypotenuse of a right-angled triangle when the sides containing the right angle are (i) 7 in., 24 in., (ii) 2.34 in., 4.65 in., (iii) 65 mm., 83.5 mm., (iv) a mm., b mm.

3. Find the remaining side and the area of a right-angled triangle, given the hypotenuse and one side, as follows: (i) Hyp. = 10 cm., side = 6 cm., (ii) hyp. = 6 in., side = 4 in., (iii) hyp. = 143 mm., side = 71.5 mm., (iv) hyp. = x in., side = y in.

4. Find the diagonal of a rectangle whose sides are (i) 20 ft. and 30 ft., (ii) a cm. and b cm.

5. Find the remaining side and the area of a rectangle, given (i) diagonal = 2.5 in., one side = 2 in., (ii) diagonal = x in., one side = y in.

6. Find the diagonal of a square whose side is (i) 15 mm., (ii) x in. What is the ratio of the diagonal to the side?

7. Find the side and area of a square whose diagonal is (i) 10 in., (ii) x ft.

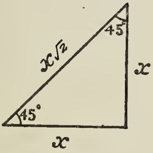


Fig. 2.4

8. Figs. 4 and 5 show the 45°—45°—90° and the 30°—60°—90° triangles. Verify that the sides are related as in the figures. These side lengths should be learned.

9. Find how far a traveller is from his starting point after the following journeys: (i) first 10 miles N., then 8 miles E., (ii) first 8 miles E., then 10 miles N., (iii) 43 km. S.W. and 32 km. S.E., (iv) 24 miles S., 10 miles E., 14 miles N. (try to complete a right-angled tri-

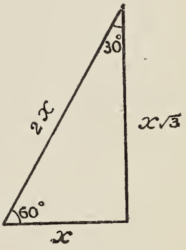


Fig. 25

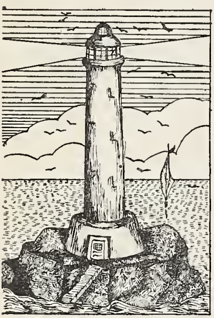
angle having the required line for hypotenuse), (v) 4 miles W., 6 miles S., 3 miles W., 1 mile N.

10. A ladder 30 ft. long rests on the top of a wall 12 ft. high with its foot on the ground 10 ft. from the wall. What length of ladder projects over the wall?

11. A gun on shore has a range of 6 miles and a vessel steers a course parallel to the shore so as to pass within $2\frac{1}{2}$ miles of the gun. Over what distance is the vessel within range of the gun?

12. A, B, C are three towns in a straight line running east to west. P is another town due N. of B. If $PA = 8$ miles, $PB = 6$ miles, $BC = 4.5$ miles, find the distances PC and AB.

13. A mountain railway up a straight incline crosses the 200 and 250 ft. contours at points 500 yd. distance apart. Find by calculation the gradient in the form 1 in n, where n is measured along the horizontal.



14. A roof truss is constructed as follows: AB is horizontal and 24 ft. long, CD is vertical from C, the mid-point of AB, and 12 ft. long. CE and CF are perpendicular to AD and DB. Calculate the lengths of the members AD and CE.

15. A lighthouse shows a light through a sector from E. to S.E. Steering N.E. the captain of a 10-knot vessel first sees the light at 1 a.m. At 1.30 a.m. the light disappears. How far off is it then?

Three Dimensions

16. A block of metal in the shape of a cube contains a cubic decimetre. Find the lengths of the face diagonals and the main diagonals of the cube.

17. A cuboid (rectangular box) has dimensions 18 in. x 14 in. x 10 in. Find the lengths of its face diagonals and its main diagonals.

18. Fig. 6 shows one of the hoppers of a gravel crushing machine. The upper rectangle has dimensions 12 ft. x 8 ft., while those of the lower rectangle are 5 ft. x 3 ft. The depth of the hopper is 6 ft. Find: (i) the length of the slant edge; (ii) the area of each side of the hopper.

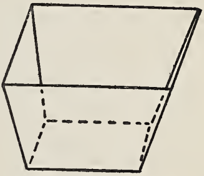


Fig. 2.6

Summary

A right triangle is determined when:

(i) *any two sides are given:*
(ii) *one side, one acute angle are given.*

The lengths of the sides of a right triangle satisfy the relation:

$$t^2 = r^2 + s^2$$

where t is the hypotenuse.

These facts will be incorporated formally later in the theory of triangles.

CHAPTER III

TRIANGLE MEASUREMENT

1. Facts concerning triangles.

You should recall all the things you know about the sides and angles of triangles. The following suggestions will help you to do this:

Types of triangles:

(a) Angle classification: acute, right, obtuse;
(b) Side classification: isosceles, isosceles-right, equilateral, scalene.

Terms in common use:

Angle and opposite side; angle included between two sides; side included between two angles; exterior and interior angles; congruent triangles.

Facts:

Facts about the angles of a triangle; about the sides of a triangle; about isosceles, equilateral, and right triangles; about pairs of congruent triangles.

When you have reviewed these ideas try the following quiz.

QUIZ ON TRIANGLES

Is it true or is it false that

1. The acute angles of an isosceles right triangle are equal?
2. Equilateral triangles are always congruent?
3. A triangle can always be made with three sticks?

4. Congruent triangles have the same area?

5. The longest side in a scalene triangle is shorter than the sum of the other two sides?

6. A hypotenuse is what is assumed in a problem?

7. The supplement of any angle in a triangle is equal to the sum of the other two angles of the triangle?

8. If an acute angle in a right triangle is doubled, the side opposite this angle is also doubled?

9. The largest angle in a triangle is less than the sum of the other two angles?

10. A triangle with two (or three) equal angles has two (or three) equal sides?

11. The longest side of an isosceles right triangle is 3/2 either of the other equal sides?

12. When the sum of two angles of a triangle is given, the third angle can be found?

13. A triangle can always be drawn with two of its angles 65° and 50° and one side any assigned length?

14. The length of the altitude of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times the length of the side of the triangle?

15. Two sides and one angle fix a triangle?

2. The specifications for a triangle.

The following sets of specifications will fix a triangle:

(i) Two sides and their included angle (S. A. S.);
(ii) Two angles and one side (A. S. A.);
(iii) Three sides (S. S. S.).

It is understood that the parts (sides or angles) must meet certain requirements for a triangle, viz., (i) and (ii), an angle or the sum of two angles must be <180°; (iii), no side may be greater than the sum of the other two.

EXERCISES. Triangle Specifications

1. Construct triangles to the following measurements; in each case measure the remaining parts:—

(i) BC $=3.18$ in., AB $=3.18$ in., $\angle B=33.5^\circ$;
(ii) $a=2.39$ in., $b=2.44$ in., $\angle C=63.5^\circ$;
✓ (iii) AB $=2.82$ in., AC $=2.77$ in., $\angle A=137^\circ$;
✓ (iv) $c=3.00$ in., $\angle A=61^\circ$, $\angle B=59^\circ$;
(v) BC $=3.52$ in., $\angle B=25^\circ$, $\angle C=23^\circ$;
(vi) AC $=10.65$ cm., $\angle A=54.5^\circ$, $\angle C=36^\circ$;
(vii) BC $=6.40$ cm., CA $=9.05$ cm., AB $=7.63$ cm.;
(viii) $a=7.69$ cm., $b=9.30$ cm., $c=5.30$ cm.;
(ix) AB $=2.99$ in., $\angle B=127.5^\circ$, $\angle C=53.5^\circ$;
(x) $c=2.92$ in., B $=59^\circ$, $b=2.39$ in.;
(xi) $\angle B=33.5^\circ$, BC $=2.61$ in., CA $=1.54$ in.;
✓ (xii) $a=2.16$ in., $b=2.64$ in., $\angle B=64.5^\circ$.

2. Try to construct the following triangles RST:

(i) $r=4.9$ in., $s=5.8$ in., $t=11.3$ in.;
(ii) R $=40^\circ$, $s=4.3$ in., $r=2.5$ in.;
✓ (iii) T $=120^\circ$, $r=760$ yd., $t=680$ yd.

3. Draw two triangles on a base of 3 in. with the other sides 2.7 in. and 2.2 in. Are these triangles congruent?

4. Draw two triangles given A $=30^\circ$, $a=5.5$ cm., $c=8.8$ cm. Are these triangles congruent?

5. A man standing on the bank of a river sees a tree on the far bank in a direction 20°W. of N. He walks 200 yards along the bank and finds that its direction is now N.E. If the river flows east and west, find its breadth.

6. A and B are two points on a straight road which runs due east and west; from A a flagstaff F bears N. 30°E., and from B it bears N. 40°W. If the distance AB is 1125 yards, find the distance of F from the road.

7. An Admiral signals to his cruiser squadron (bearing N. 40°W. 50 miles from him) to meet him at a place N. 50°E., 70 miles from his present position. Find bearing and distance of the meeting place from the cruisers.

8. **A** and **B** are two buoys 800 yards apart, **B** due N. of **A**. A vessel passes close to **B** and, steering due E., observes that after 5 minutes the bearing of **A** is 57°W. of S. Find the distance the vessel has moved.

9. A straight sand bank, 1500 yards long, runs due E. and W., and has a buoy at each end. From a ship the eastward buoy bears N. 41°E., and the bank subtends an angle of 72°. Find her distance from the bank. Explain your construction.

10. A line of telegraph poles runs 8° East of North, and the poles are 150 feet apart. Standing at one of these poles **A**, an observer determines by compass that a certain building lies 35° East of North; on proceeding northwards to the next pole **B**, the observer finds that the building lies 41° East of North. Determine the distance of the building from the first pole **A**.

11. From a ship at sea the bearings of two points on the coast are found to be S. 14° E. and S. 49° W. If one of these points is found from the chart to be 10.4 miles N. 61° W. of the other, find, by drawing to scale, the distance of the ship from each point.

12. The city of Honolulu lies on the island of Oahu. The bearing of the northernmost tip (Upolu Pt.) of the island of Hawaii from Honolulu is S. 60° 45′ E. and its distance is 147 mi. The southern tip of the island is Ka Lae. Its bearing and distance from Honolulu are S. 40° 30′ E. and 216 mi. Find the length of Hawaii.

13. From a ship at sea the following observations are made:—Dover bears N. 16° E., and Boulogne S. 81° E. From the chart it is found that Dover is 26 miles N. 24° W. of Boulogne. Find the distance of the ship from Dover.

14. **O** and **P** are points on a straight stretch of shore. **P** is 4.5 miles N. 74° E. of **O**. From **O** a ship at sea bears S. 58° E., and from **P** the ship bears S. 32° W. Find the distance of the ship from **P**, and also its distance from the nearest point of the shore.

15. The position of an inaccessible point **C** is required. From **A** and **B**, the ends of a base line 200 yards long, the following bearings are taken:—

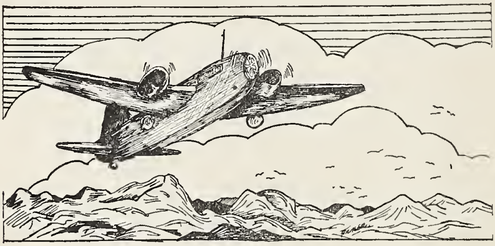
From **A**: {Bearing of **B** is N. 70° 30′ E.
" " **C** is N. 30° 20′ E.
From **B**: " " **C** is N. 59° 40′ W.

Find the distances of **C** from **A** and **B**.

3. Map exercises.

The exercises below refer to the air navigation map inserted at the end of the book.*

On such a map it is legitimate to read distance to scale and angle with protractor; the exercises are to be worked in this manner.



True north is shown on the map with an arrow. True north-south lines are lines of constant longitude, the *meridians of longitude.* True east-west lines are lines of constant latitude, the *parallels of latitude.* These are shown on the map. Certain of these lines have been subdivided for convenience to show minutes of latitude and of longitude (see ex. 18-22).

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*The outline map was prepared from the official air navigation map for the *Cranbrook-Lethbridge* region. This map may be obtained from *The Hydrographic and Map Service,* Department of Mines and Resources, Ottawa, at a cost of twenty-five cents. Similar maps for other localities are available.

Magnetic north is the direction of the compass needle. This direction is east of true north about 23° for this locality. It is not, however, the same over the region; the lines of constant *magnetic variation* (from true north) are shown on the map.

In the first two groups of exercises *bearing* will be shown thus: N. 30° E.; S. 50° W. In the last two groups of exercises a different method to be explained later will be used.

You should note that when a town is shown with an aerodrome, distance and direction are to be measured *from the aerodrome.*

Distance and True Bearing

The true bearing is to be shown: N. *a*° E., etc.

1. Find the true bearing from Cowley of (1) Macleod; (2) Pincher Creek; (3) Blairmore.

2. Find the bearing of Cardston from (1) Lethbridge; (2) Macleod; (3) Cowley.

3. A plane is half-way between Cranbrook and Fernie. What is the bearing of Tornado Mountain from the plane?

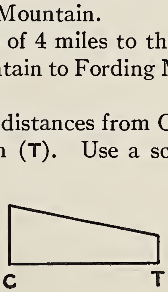
4. (a) Find the air distance from (i) Cranbrook to Fernie; (ii) Fernie to Coleman.

(b) Find the distance along the highway for (i) and (ii) in (a).

5. (a) Find the distances and bearings of Fording Mountain and Crowsnest Mountain from Thunder Mountain.

(b) Draw the data in (a) to a scale of 4 miles to the inch and find the distance from Crowsnest Mountain to Fording Mountain. Check on the map.

(c) Draw a line and mark on it the distances from Crowsnest Mountain (**C**) to Thunder Mountain (**T**). Use a scale of 2 miles to the inch. Use the elevations of these mountains as given on the map to measure the distance from the top of one mountain to the top of the other, as suggested in the figure. Compare with **CT**.



Distance and Magnetic Bearing

Magnetic and true bearings are found one from the other by means of the magnetic variation. Use a variation of N. 23° E. *for all points west of the line running midway between the lines of magnetic variation marked* 22° E. *and* 23° E. *Otherwise use a variation of* N. 22° E.

EXAMPLES. *The true bearing of The Steeples from Cranbrook is* N. 79° E., *and hence the magnetic bearing of The Steeples from Cranbrook is* N. 56° E.

The true bearing of Pincher Creek from Magrath is N. 85° W. *and hence the magnetic bearing of Pincher Creek from Magrath is* S. 73° W.

6. Find the magnetic bearing of each of the courses, the true bearing being given. Draw freehand sketches to illustrate.

(a) Cardston to Magrath; N. 51° E.;
(b) Cardston to Pincher Creek; N. 55° W.;
(c) Pincher Creek to Raymond; S. 89° E.;
(d) Raymond to Lethbridge; N. 28° W.

7. A plane flies from Lethbridge to Waterton Park. Find directly from the map the magnetic bearing of this course from Lethbridge. To do this draw the line on the map joining the two places. Then draw a line through Lethbridge N. 22° E., and measure the angle between the two lines. What is the magnetic bearing for the return flight from Waterton Park?

8. As in ex. 7, find directly from the map the compass bearing of

(a) Coleman from (i) Fernie, (ii) Packhorse Peak;
(b) Cranbrook from (i) Kimberley, (ii) The Steeples.

Distance and Bearing

In air navigation bearing is measured from the north clockwise to the required course. Thus, on the map, the magnetic bearing east out of Cowley is 63° M., *which is measured clockwise from magnetic north. The bearing going west into Cowley is* 243° M., *that is* 243° *clockwise from magnetic north. The true bearings of these courses from Cowley are* 86° T. *and* 266° T.*

9. Draw on the map the direction of magnetic north through (a) Cranbrook, (b) Cowley, (c) Lethbridge. Then check by measurement the various directions indicated on the map for the radio ranges. Express these in terms of true north. Check some of these true bearings on the map.

10. "When the magnetic variation is easterly it is subtracted from the true course to obtain the magnetic course." Do you agree with this rule?

11. Find the distance, the magnetic and true bearing of Vimy Peak from Mt. Thompson-Seton. Sketch to a scale of 16 miles to the inch, showing true and magnetic north on the sketch. What is the magnetic bearing of Mt. Thompson-Seton from Vimy Peak?

12. As in ex. 11 for the bearing of Cranbrook from Newgate and of Newgate from Cranbrook.

13. What is a rule by which the magnetic bearing of A from B can be obtained from the bearing of B from A? Check your rule when the bearing of B from A is 40° M.

14. Draw the complete course from Lethbridge to Macleod to Pincher Creek to Waterton Park to Lethbridge, scale 16 miles to the inch, and relative to magnetic north. Label each leg of the course with its length and its magnetic bearing.†

---

*The use of the clockwise direction to measure positive angle is confined to navigation. In other scientific work where signed angle is necessary, the direction of positive angle is *counter-clockwise.*

†The directions of the course may be checked with Theorem 17 (chap. 6).

15. (a) A plane flying east from Cranbrook is off the beam and hears the signal N. Is its bearing greater or less than the beam direction 64° M.?

(b) A plane flying north into Cowley is off the beam and hears the signal A. Is its bearing greater or less than the beam direction 338° M.?

✓**16.** Three stations X, Y, Z have magnetic bearings as follows: X to Y, 210° M.; Y to Z, 150° M.; Z to X, 350° M. Use a sketch to find the angles of ΔXYZ.

17. Jim said, "If you want the angle between two directions, add 180° to the second bearing and subtract the first bearing from this sum." Jim then found that his rule was sometimes wrong. Can you amend Jim's rule to make it work for any two directions?

Latitude and Longitude

The latitude and longitude fix the position of a point on the map. On a large-scale map the parallels of latitude and the meridians of longitude form a rectangular co-ordinate system. Position will be shown as (Lat. 49° 12′ N., Long. 113° 42′ W.). *The parallel of latitude* 49° 30′ N. *on the map is graduated in minutes of longitude, while various meridians are graduated in minutes of latitude.*

18. Find the positions of (a) the aerodrome at Cowley; (b) the golf course near Newgate; (c) Mt. Ptolemy; (d) Magrath.

19. The bearing of a plane from Cowley is 50° M. and from Lethbridge is 279° M. Find its position.

20. A plane leaves Cowley at 200 m.p.h. Six minutes later its bearing from Lethbridge is 230° M. Find the two possible positions of the plane.

21. A plane bears east from Mt. Erickson and 132° T. from Tornado Mountain. A few minutes later its bearing from Erickson is 71° T. and from Tornado is 100° T.

(a) Plot the course of the plane on the map and find the direction of the course;

(b) What is the latitude of the plane when it crosses the Fifth meridian (114° W.)?

22. An alpine party is lost. A plane sees their signal and photographs their position, which was shown from an aerial map to be (Lat. 49° 17′ N., Long. 114° 45′ W.). Supplies are flown in from Cranbrook. What was the direction of the supply plane out of Cranbrook? What is the bearing of the position from Cowley?

CHAPTER IV

THE TRIANGLE. SIDE AND ANGLE PROPERTIES

1. Exercises.

The facts about triangles are summarized at the end of the chapter. The exercises of this paragraph are based on the theory to the end of Theorem 13. These theorems are probably familiar. If they are, begin working the exercises. Thus exercises 1-8 assume Theorem 6 and 7 to be true, and of course any of the earlier Theorems 1-5. If any one of Theorems 6-13 is not familiar you should try to prove it, unless it is to be assumed (‡). If you fail to get a proof, consult the hints given in Chap. X.

The following examples illustrate a mode of proof which is often useful. You will want to try this scheme in the exercises.

Example 1

XYZ *is an isosceles triangle with* XY = XZ. *The bisectors of* ∠Y *and* ∠Z *meet at* O. *Prove that* OX *bisects* ∠X.

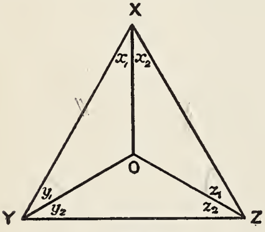


Fig. 4.1

Data Δ XYZ.

(1) XY = XZ

(2) $\angle y_1 = \angle y_2$ and $\angle z_1 = \angle z_2$.

To prove that (i) $\angle x_1 = \angle x_2$.

Discussion. (a) Assume $\angle x_1 = \angle x_2$. Then you can prove that ΔOXY ≡ ΔOXZ and hence that OY = OZ. How?

(b) If you can show from the data that OY = OZ then you can prove that ΔOXY ≡ ΔOXZ and hence that $\angle x_1 = \angle x_2$. Can you see how to show that OY = OZ? (Hint: Use Theorem 12.)

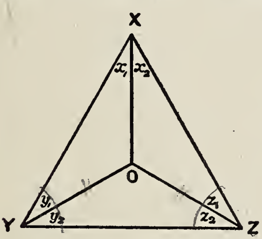
The method of the above example consists in assuming as true what is to be proved; then to see what happens. Why was the word *data* used? What are the data in any problem?

The proof for ex. 1 is shown below. This will serve as a model statement of the proof of a proposition. You should note the main paragraphs in the statement. (i) Enunciation of the proposition; (ii) figure; (iii) data; (given things; hypotheses); (iv) requirements (things to be proved); (v) construction; (vi) proof; (vii) conclusions (findings).

It should be emphasized that (v)—(vii) are based on reasoning not shown in the final statement of a proof (see "Discussion" in the above examples). You may well include a paragraph after (iv) showing the reasoning which led to (v)—(vii).

Proof of Example 1

XYZ *is an isosceles triangle with* XY = XZ. *The bisectors of* ∠ Y *and* ∠ Z *meet at* O. *Prove that* OX *bisects* ∠ X.



Data Δ XYZ.

(1) XY = XZ,

(2) $\angle y_1 = \angle y_2$ and $\angle z_1 = \angle z_2$.

To prove that (i) $\angle x_1 = \angle x_2$.

Construction Join OX.

Proof In Δ XYZ, XY = XZ (Data 1),

Hence ∠ Y = ∠ Z. (Isosc. Δ, equal sides) (Theorem 12,)*

∴ $\angle y_1 = \angle z_1$ and $\angle y_2 = \angle z_2$ (Data (2)).

In Δ OYZ, $\angle y_2 = \angle z_2$,

∴ OY = OZ. (Isosc. Δ, equal angles) (Theorem 13.)

In Δs OXY, OXZ

∵ { XY = XZ, (Data (2))
OX is common,
OY = OZ,

∴ $\angle x_1 = \angle x_2$. (S.S.S.)(Theorem 10.)

Hence OX bisects ∠ X.

---

*A clear reference should be made to the property being applied. The theorem number is added for completeness.

Example 2

ABCD *is a quadrilateral with its opposite sides parallel, i.e., a parallelogram. Prove that* **AB = CD, AD = BC.** (*See ex. 22* (*i*) *in the exercises.*)

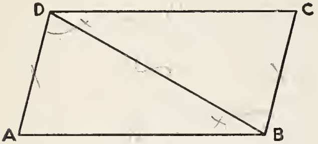


Fig. 4.2

Data Quadrilateral **ABCD**
(1) **AB** is ∥ to **CD**
(2) **AD** is ∥ to **BC.**

To prove that
(i) **AB = CD**
(ii) **AD = BC.**

Discussion. Assume (i) and (ii) to be true. Join **BD.** Then Δ **ABD** ≡ Δ **BDC.** Why? If you can prove Δ **ABD** ≡ Δ **BCD** from (1) and (2), then (i) and (ii) will follow.

Write out the proof for example 2.

EXERCISES

Angle Properties (Theorems 6, 7)

1. In fig. 3, which are the interior angles opposite to (i) ∠**ABD**, (ii) ∠**ACE**? In fig. 4, which are the interior angles opposite to (i) ∠**BCY**, (ii) ∠**CBX**?

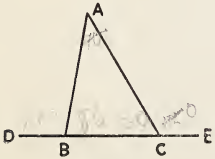


Fig. 4.3

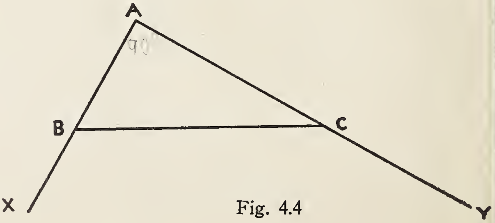


Fig. 4.4

2. In fig. 3 ∠A = 40°, ∠ACE = 120°; find the remaining angles.

3. In fig. 4 ∠A = 90°; prove that ∠XBC + ∠YCB = 3 rt. ∠s.

4. Two straight by-roads cross a straight main road at different points and make angles of 60° and 42° with it. At what angle do the by-roads meet?

5. *Prove that, if the sides of a triangle are produced in order, the sum of the three exterior angles is four right angles.*

6. *Draw a triangle* ABC *with* AB *greater than* AC. D *is a point in* AB *such that* ∠ACD = ∠ADC. *Prove that* ∠ACB *is greater than the* ∠ABC. (Theorem 14.)

7. If O is any point inside a ΔABC, prove ∠BOC greater than ∠BAC. (Produce BO to cut AC in D.)

8. A triangle has the two base angles equal, and a side is produced beyond the vertex. Prove that the exterior angle is twice a base angle.

Isosceles Triangles (Theorems 12, 13)

9. Prove that if a triangle PQR is equiangular, it must also be equilateral. (∠Q = ∠R, ∴ side? = side?.) State and prove the converse.

10. Through the vertex P of an isosceles triangle PQR a straight line XPY is drawn parallel to QR; prove that ∠QPX = ∠RPY.

11. In fig. 5 there is an isosceles triangle with a line drawn parallel to the base. Assuming the vertical angle to be 40°, find all the other angles in the figure. Make a freehand sketch showing all angles.

12. Prove that a straight line drawn parallel to the base of an isosceles triangle to cut the equal sides forms another isosceles triangle. (See fig. 5.)

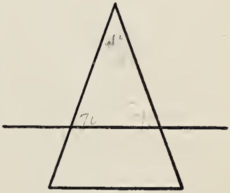


Fig. 4.5

13. XYZ is an isosceles triangle; the bisectors of the equal angles (Y, Z) meet at O; prove that $\triangle OYZ$ is also isosceles.

14. From Q and R, the extremities of the base of an isosceles triangle PQR, perpendiculars are drawn to the opposite sides. If these perpendiculars intersect at X, prove that $XQ = XR$.

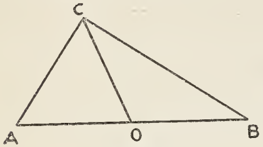


Fig. 4.6

15. From the mid-point O of a straight line AB a straight line OC is drawn; if $OC = OA$, prove that $\angle ACB$ is a right angle. (See fig. 6.)

16. If through any point in the bisector of an angle a line is drawn parallel to either of the arms of the angle, prove that the triangle thus formed is isosceles. (See fig. 7.)

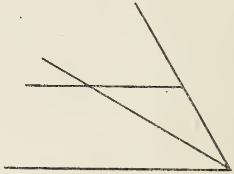


Fig. 4.7

17. Prove that the angle between a diagonal and a side of a square is 45°.

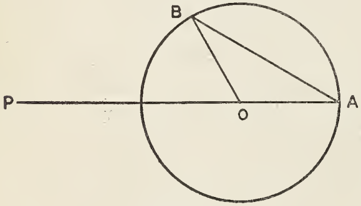


Fig. 4.8

18. OA, OB are radii of a circle, AO is produced to P; prove that $\angle BOP = 2\angle BAP$. (See fig. 8.)

19. If the $\angle$s G, H of a triangle FGH are each twice $\angle F$, and if the bisector of $\angle G$ meets FH in K, prove that $FK = GK$ and $GK = GH$.

20. ABC is a triangle, BA is produced to X so that $AX = AC$, CA is produced to Y so that $AY = AB$. Prove that BY is parallel to CX.

Congruence (Theorems 8-11)

21. **ABC** and **DEF** are two triangles. If the following facts hold, are the triangles necessarily congruent (give reasons for your answers):

(i) $AB = DE$, $AC = DF$, $\angle A = \angle D$; Ye
(ii) $AB = DF$, $AC = DE$, $\angle A = \angle D$; No
(iii) $AB = EF$, $AC = DF$, $\angle A = \angle F$; N
(iv) $BC = DE$, $\angle A = \angle F$, $\angle B = \angle D$; No
(v) $BC = DE$, $\angle A = \angle E$, $\angle B = \angle F$? No

22. **ABCD** is a quadrilateral with its opposite sides parallel (i.e., a parallelogram). Prove that:

(i) $AB = CD$, $AD = BC$;
(ii) $\angle A = \angle C$ and $\angle B = \angle D$;
(iii) $AO = OC$, $BO = OD$, where O is the point of intersection of the diagonals.

23. In fig. 9, **AB** and **CD** are equal and parallel; prove that $AC = BD$.

24. Show that, if the diagonals of a quadrilateral bisect one another, opposite sides are parallel.

25. The equal sides **QP**, **RP** of an isosceles triangle **PQR** are produced to **S**, **T** so that $PS = PT$; prove that $TQ = SR$.

26. The equal sides **AB**, **AC** of an isosceles triangle **ABC** are produced to **X**, **Y** so that $AX = AY$. Prove that $CX = BY$.

27. **AB** is equal and parallel to **CD** (see fig. 9). Show that **AD** bisects and is bisected by **BC**.

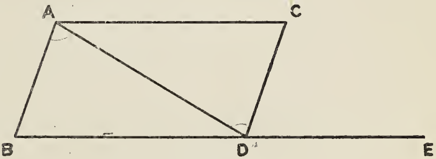


Fig. 4.9

28. **GHK** is an equilateral triangle; in **GH**, **HK**, **KG** take points **X**, **Y**, **Z** so that $GX = HY = KZ$. Prove that the triangle **XYZ** is equilateral.

29. If the diagonal PR of a quadrilateral PQRS bisects the angles at P and R, prove that the quadrilateral has two pairs of equal sides.

30. A triangle ABC has $AB = AC$; prove that the perpendiculars from B and C on the opposite sides are equal to one another.

31. In fig. 10, PM, QN are drawn perpendicular to the diameter AOB, O being the centre of the circle; show that, if $PM = QN$, then $\angle POM = \angle QON$.

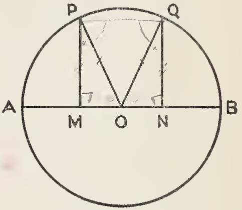


Fig. 4.10

✓ **32.** A triangle XYZ has $\angle Y = \angle Z$; prove that the perpendiculars from the mid-point of YZ to XY and XZ are equal to one another.

✓ **33.** If the perpendiculars from two corners of a triangle to the opposite sides are equal, the triangle is isosceles.

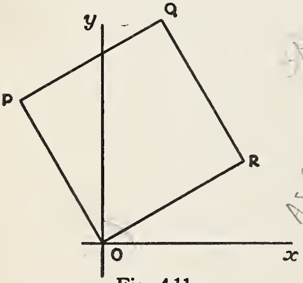


Fig. 4.11

34. ABCD is a square; E, F, G are the mid-points of AB, BC, CD respectively. Join EF and FG and prove them equal.

35. In fig. 11, OX, OY are a pair of axes at right angles to one another and OPQR is a square. Prove that the distance of P from Oy is equal to the distance of R from Ox.

2. The congruence theorems.

You will have noticed that the theorems about the congruence of triangles deal with three groupings of sides and angles:

(a) Three sides are given; (S.S.S.)

(b) Two sides and one angle, the angle included between the sides, are given; (S.A.S.)

(c) One side and two angles are given. (A.S.A.)

The case of two sides and one angle, the angle not included between the sides, has already been encountered in the exercises. In this instance no congruence theorem is possible. Why? What of the case when three angles are given?

3. Inequalities.

The proofs of Theorems 14 and 15 are outlined below. You should work through these proofs in detail.

Theorem 14. **If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.** (See ex. 6, p. 35.)

Data $\triangle ABC$.
$c > b$.

To prove that $\angle C > \angle B$.

Discussion. The fact that $c > b$ suggests marking D at distance b from A.

(a) $\angle x = \angle y$;
(b) $\angle x = \angle B + \angle z$;
(c) $\angle C = \angle y + \angle z = \angle x + \angle z$.
(d) $\angle x > \angle B$;
(e) $\angle C > \angle x > \angle B$.

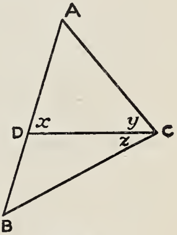


Fig. 4.12

Note the argument by emphasis here. $\angle C > \angle x$. If $\angle x > \angle B$, then how much more is $\angle C > \angle B$?

Theorem 15. (Converse of Theorem 14.) **If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.**

Data $\triangle ABC$.
$\angle C > \angle B$.

To prove that $c > b$.

Discussion. There are three possibilities regarding b and c. (i) $b=c$; (ii) $b>c$; (iii) $b<c$. Show that only one of these possibilities can be realized.

Here is the method of elimination again, this time with three possibilities present.

EXERCISES. Inequalities

1. Prove Theorem 16 (see fig. 13). Hint: Prove that $AD > AB$ and hence that $a+b>c$. Have you proved that $a+c>b$ or that $b+c>a$?

2. Prove Theorem 15 by assuming Theorem 16 to be true.

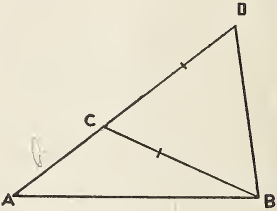


Fig. 4.13

✓ **3.** In fig. 14 as P moves around the circle from B to B', show that AP always increases, i.e., $AP' > AP$. (Hint: Let AP' meet the bisector of $\angle POP'$ in V. Join PV.) Note how this property is fundamental to the alternating motion of the piston in fig. 15.

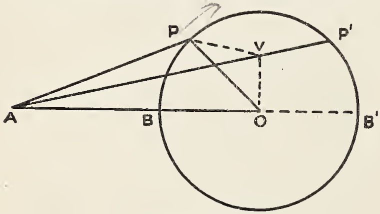


Fig. 4.14

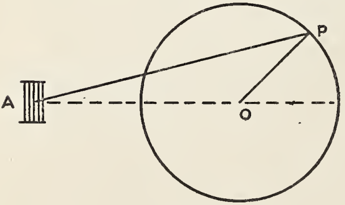


Fig. 4.15

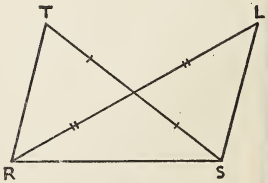


Fig. 4.16

4. Prove that the exterior angle of a triangle is greater than either of the interior opposite angles by (i) Theorem 7; (ii) the congruence of triangles as suggested in fig. 16.

5. Argument by emphasis:

Tom (discouraged): "I can't play football, Dad."
Dad: "Who else is playing, Tom?"
Tom mentions that Jack Kerr plays.
Mother: "Why, Tom Rogers, Jack is two years younger than you are, and he is smaller, too. If he can play, so can you!"

The mother's reasoning may be re-stated as follows: (1) Jack can play football; (2) Tom is larger and older than Jack. Therefore (3) Tom is capable of playing football. Is this sound reasoning? Give reasons for your criticism.

6. Mary: "I don't see why we can't have a new car. The Dingles have one, and well——"
Mother: "You mean, Mary, that we are at least as well off as the Dingle family?"
Mary: "Yes, I do. If they have a new car, why can't we have one?

State Mary's argument in three steps as in ex. 5. Is the argument a reasonable one?

7. The method of elimination. This method is a common mode of argument in mystery stories. The principle is clearly stated in Sherlock Holmes, *The Sign of Four.**

> "How came he then?" I (Watson) reiterated. "The door is locked; the window is inaccessible. Was it through the chimney?"
>
> "The grate is much too small," he answered. "I had already considered that possibility."
>
> "How, then?" I persisted.
>
> "You will not apply my precept," he said, shaking his head. "How often have I said to you that when you

---

*Conan Doyle, *The Complete Sherlock Holmes* (Doubleday Doran, 1930), vol. 1, pp. 45-46.

have eliminated the impossible, whatever remains, *however improbable*, must be the truth? We know that he did not come through the door, the window or the chimney. We also know that he could not have been concealed in the room, as there is no concealment possible. How, then, did he come?"

"He came through the hole in the roof!" I cried.

"Of course he did. He must have done so. If you will have the kindness to hold the lamp for me, we shall now extend our researches to the room above—the secret room in which the treasure was found."

(a) List the various possible ways in which the intruder could have entered the room. Number these possibilities (1), (2), etc.

(b) Opposite each possibility write the reason given for its elimination.

(c) Is it certain that the intruder came through the hole in the roof?

8. Sherlock Holmes, *The Sign of Four*.*

"That other man again!"

"I have no wish to make a mystery of him to you, anyway. But you must have formed your own opinion. Now, do consider the data. Diminutive footmarks, toes never fettered by boots, naked feet, stone-headed wooden mace, great agility, small poisoned darts. What do you make of all this?"

"A savage!" I exclaimed. "Perhaps one of those Indians (native of India) who were associates of Jonathan Small."

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*Ibid. p. 75.

> "Hardly that," said he (Holmes). "When first I saw signs of strange weapons I was inclined to think so, but the remarkable character of the footmarks caused me to reconsider my views. Some of the inhabitants of the Indian Peninsula are small men, but none could have left such marks as that. The Hindoo proper has long and thin feet. The sandal-wearing Mohammedan has the great toe well separated from the others because the thong is commonly passed between them. These little darts, too, could only be shot in one way. They are from a blow-pipe. Now, then, where are we to find our savage?"

(a) Number each item in the data (1), (2), etc.

(b) Holmes argues by assuming with Watson that the man was a native of India. List the three possible types of Indians which Holmes considers.

(c) Holmes considers each type in turn. State the reasons Holmes gives for rejecting all three possibilities.

II. Side and Angle Properties of Triangles

A. ANGLE PROPERTIES

THEOREM 6. **The sum of the angles of a triangle is equal to two right angles.**

COROLLARY. **If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.**

THEOREM 7. **If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.**

B. CONGRUENT TRIANGLES

‡THEOREM 8. **If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles included by those sides equal, the triangles are congruent.** (S.A.S.)

COROLLARY. (*Right triangles*). **If two right triangles have the sides about the right angle of one triangle equal to the sides about the right angle of the other, each to each, the triangles are congruent.**

‡THEOREM 9. **If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.** (A.S.A.) or (A.A.S.)

COROLLARY. (*Right triangles*). **If two right triangles have an acute angle of the one equal to an acute angle of the other, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.**

‡THEOREM 10. **If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.** (S.S.S.)

THEOREM 11. (*Right triangles*). **If two right triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.**

C. ISOSCELES TRIANGLES

THEOREM 12. **If two sides of a triangle are equal, the angles opposite to these sides are equal.**

THEOREM 13. (CONVERSE OF THEOREM 12.) **If two angles of a triangle are equal, the sides opposite to these angles are equal.**

D. SIDE AND ANGLE INEQUALITIES

THEOREM 14. **If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.**

THEOREM 15. (CONVERSE OF THEOREM 14). **If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.**

THEOREM 16. **Any two sides of a triangle are together greater than the third side.**

CHAPTER V

CONSTRUCTIONS

1. The use of instruments.

What instruments did you use to construct (1) a right triangle; (2) any triangle according to specifications? List the instruments used in each particular case.

The constructions listed at the end of Chapter 1 should be reviewed. These should be compared with the following, which are the methods used by a draughtsman or an engineer.

Engineering Constructions

1. *To make an angle at* O *equal to an angle at* A.

(a) If the angle at O is to have its sides parallel to those of the angle at A, draw parallels as required through O (see Const. 5 below).

(b) If the angle at O is to be differently placed, a right triangle construction is used. Draw on the angle at A a right triangle with sides of good length. (See fig. 1.) Measure these lengths x and y and reconstruct the right triangle at O in its desired position. x would be taken as, say, 5 or 10 in. (See Const. 4 and 6 below.)

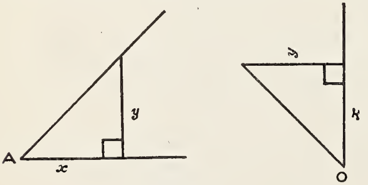

Fig. 5.1

(c) A common protractor is seldom used, being a poor practical instrument. A precision protractor with a vernier is obtainable but it is very expensive.

2. *To bisect an angle.*

The construction of chapter 1 is used, but with dividers serving in the place of compasses, for greater accuracy.

3. *To draw the perpendicular bisector of a straight line segment.*

This construction is really a double one:

(i) a bisection: (ii) the erection of a perpendicular.

(i) The mid-point is guessed. Dividers are then set with the guessed half distance. This length is laid off from each end of the given segment. The required mid-point is thus the mid-point of a very short segment and hence can be located accurately by inspection.

(ii) The perpendicular at the mid-point is drawn with a set square. If greater accuracy is required, circle arcs are used as in the theoretical construction.

4. *To draw a perpendicular through a point to a straight line.*

As in construction 3(ii).

5. *To draw a straight line through a point parallel to a straight line.*

A pair of set squares is usually used, unless there is available a precision parallel ruler. Such a ruler is of steel mounted on end rollers which have a common axle.

This instrument although superior to the use of set squares, is expensive, and consequently for the individual worker the set square method is the common one.

In addition to the above constructions there is another, which, owing to its difficulty (see §2, ex. 20) has been omitted until now. This construction is the following:

6. *From a given point to draw a straight line segment equal to a given straight line segment.*

In draughting this transfer of length is best carried out directly with a fine pencil or needle point and an accurately graduated scale. Transfer by dividers is less accurate.

Mention should be made of the use of the T-square. In building construction design particularly, many "horizontal" and "vertical" lines are needed. The drawing sheet is placed so that the T-square rulings are parallel to the edges of the sheet.

Such lines are also used in the mapping of "traverses", stations being located by means of "latitudes and departures". In this work of triangulation, the protractor is rarely used. The protractor may be used, however, to locate isolated points on the map, which do not serve as reference points for further measurement.

You should repeat the above constructions a number of times with the instruments you have available.

In problem work the figures are, of course, usually drawn freehand. When scale figures are required, the above methods may be freely used, unless instructions are given to the contrary.

2. The limited use of instruments.

Every game is played according to certain rules. The rules may be changed; the game then takes on a different character. The introduction of the forward pass changed Canadian rugby. You should find other examples of games in which rules have been modified.

For the Greek thinkers geometry was largely a game, and the game was played for centuries according to two rules, viz., only two instruments were permitted: (1) an ungraduated straight edge; (2) a pair of collapsing compasses.

These restrictions led to difficulties.* Thus it was early known that no construction was possible for the trisection of an angle (other than special angles like 90°) under the above rules. Because of these difficulties, the Greeks extended their study to more complex figures, of which the most important were the sections of a circular cone made by a plane. Three new curves were found, the *parabola*, the *ellipse* and the *hyperbola*. Indeed, these plane curves, the so-called conic sections, were invented as a means of solving the problem: To construct a cube whose volume is proportional to that of a given cube. This problem was impossible of solution under the rules stated above.

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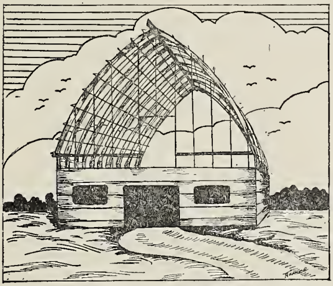
*"No inquiry can ever get under way until and unless some difficulty is encountered in a given situation." Find some examples to illustrate this proposition.



Barn Construction Parabolic Roof

The Greeks studied the conics for their properties, with no thought of application. It was not until modern times that applications were made, when *Kepler* and *Newton* found conic properties of immediate value in their studies of planetary motion. Since then they have been applied to many problems in engineering design.

EXERCISES. Constructions

The only instruments to be used in these exercises are a straightedge and compasses. Some of these constructions may suggest possible new theorems. You should state these theorems, and perhaps try to prove some of them.

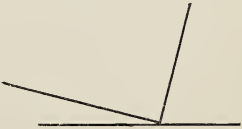


Fig. 5.2

1. Draw a triangle ABC; at a point O construct the angles of the triangle in the manner of fig. 2.

2. Draw a straight line EF and mark a point G (about 2 in. from the line); through G construct a

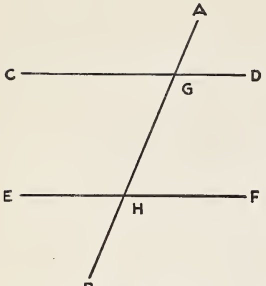


Fig. 5.3

line parallel to EF. (Draw any line through G cutting EF at H; make $\angle HGC = \angle GHF$; see fig. 3.)

3. Draw a large pentagon. Construct an equiangular pentagon with its sides half as long. Measure a pair of corresponding diagonals in each.

4. Draw two lines AOA′, BOB′, intersecting in O. Bisect the four angles by lines OP, OQ, OR, OS. (OP, OR in vertically opposite angles). What fact is suggested about (1) OP and OR, (2) OP and OQ?

5. Draw a large triangle and bisect each of its angles. What fact is suggested?

6. Draw a triangle ABC and produce the sides AB and AC. Bisect the $\angle A$ and the exterior angles at B and C (draw long lines). What fact is suggested?

7. Draw two parallel lines and a line crossing them. Bisect all four interior angles. What figure is formed by the bisectors?

8. Draw a large acute-angled triangle; draw the perpendicular bisectors of its three sides. What fact is suggested?

9. Repeat ex. 8 for (i) a right-angled, (ii) an obtuse-angled triangle.

10. Draw any three chords of a circle and the perpendicular bisector of each. What fact is suggested?

11. Describe a circle and on it take three points A, B, C; join BC, CA, AB. Bisect angle BAC and draw the perpendicular bisector of BC. Produce the two bisectors to meet.

12. ABC is any triangle. Construct a perpendicular to AB from B and a perpendicular to AC from C. Let these lines meet in R. Verify by measuring that the angles BAC and BRC are equal or supplementary.

13. Draw a large acute-angled triangle; from each vertex draw a perpendicular to the opposite side. What fact is suggested?

14. Repeat ex. 13 with a right-angled triangle.

15. Repeat ex. 13 with an obtuse-angled triangle. (You will have to produce 2 of the sides.)

16. From the centre of a circle drop a perpendicular on to a chord of the circle. What fact is suggested?

17. Construct angles of 60°, 30°, 45°, 120°, 150°, 135°.

18. Draw, and copy on tracing paper (i) a quadrilateral, (ii) a pentagon, using intersecting arcs. If tracing paper is not used, the copy may be checked by comparing its angles with those of the original.

19. Prove each of the constructions 1-5 of chapter **I.**

20. *Construction 6.* Since the Greeks did not admit the use of a scale, distance was transferred by the following ingenious construction (Euclid, Book I, Prop. 2*):

Problem. From a given point to draw a straight line (segment) equal to a given straight line (segment).

Data. **A** is the given point. **BC** is the given segment.

To draw from **A** a segment equal to **BC.**

Construction. Join **AB.** On **AB** draw an equilateral triangle **DAB.** Produce **DB** and **DA.** With centre **B**, radius **BC** draw the circle **CGH**, meeting **DB** produced at **G.** With centre **D**, radius **DG**, draw the circle **GKL**, meeting **DA** produced at **L.** Then **AL = BC.**

Prove this construction.

**The Elements of Euclid* (Nelson, Home Univ. Lib., 1932), p. 8.

In ex. 21-32 a scale (graduated straight-edge) and compasses may be used.

21. Construct an angle AOB of 60°. P is a point on the bisector of the angle 3 in. from O. Find by construction and measurement the distance of P from each arm of the angle.

22. Verify by accurate construction and measurement that lines bisecting the angles of a parallelogram intersect so as to form a rectangle.

23. Draw a large triangle and the bisectors of two of its angles. From the point of intersection of the bisectors construct perpendiculars to each of the three sides of the triangle. Measure their lengths. State the fact suggested.

24. Verify by accurate construction that in an equilateral triangle the bisector of an angle and the perpendicular bisector of the opposite side are the same line.

25. Construct a rectangle with diagonals 3.5 in. intersecting at an angle of 30°.

26. Construct a rhombus with diagonals 8 cm. and 10 cm. Measure a side.

27. Construct a rhombus with an angle of 60° and the diagonal bisecting that angle 3 in. long.

28. Construct a parallelogram having $BC = 6$ cm., $AB = 5$ cm., and $\angle BAC = 90°$.

29. Two posts 20 ft. and 15 ft. high are placed upright in the ground 12 ft. apart. Find from an accurate drawing the length of a straight wire joining their tops.

30. Construct any triangle with angles of 30°, 60°, 90°. How do the lengths of the sides including the 60° compare? Test your conclusion by drawing another triangle of different size but with the same angles.

31. At the centre of a circle draw two lines at right angles and bisect all the angles. Join, in order, the points where all the lines meet the circle. What figure is formed?

32. Divide the angle at the centre of a circle into six equal parts each 60° and join the points, in order, where the lines cut the circle. What figure is formed?

3. The sections of a circular cone.

You saw in §2 how the rules laid down by the Greeks for their game of geometry led them into difficulty. In the attempt to solve one of the resulting problems, *Menaechmus* (375-325 B.C.), one of the brilliant scholars in Plato's academy, invented the conic sections.

The sections were made with ordinary circular cones, of three types, and plane sections were taken perpendicular to a generator of each cone, giving the sections as follows:

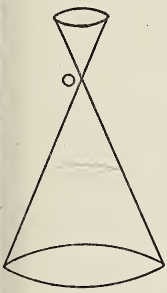


Fig. 5.4

Cone	*Section*
Acute angle	Ellipse
Right angle	Parabola
Obtuse angle	Hyperbola

During the century following *Menaechmus* the sections were made from a double cone by changing the angle at which the section was made. The double cone is needed only in the case of the hyperbola which has two parts or branches, as in fig. 4. Any plane section of both cones of the double cone is a hyperbola, save that through O, which consists of a pair of straight lines.

EXERCISES. The Conic Sections

1. Make models of the cones of *Menaechmus* and cut plane sections perpendicular to a generator of each cone. (Use modelling materials or cones turned on a wood lathe).

2. Trace the sections of ex. 1 on paper and label each section with its name and the method by which it was obtained.

3. Use a single cone as in ex. 1 to obtain the different conic sections, by varying the angle at which the section is made. The parabolic section is that one parallel to a generator of the cone.

4. As in ex. 2 for ex. 3.

Constructions

6. *From a given point to draw a straight line segment equal to a given straight line segment.*

CHAPTER VI

POLYGONS. AREA. THE PYTHAGOREAN THEOREM

1. Introduction.

The quiz which follows concerns four-sided figures. You should recall all you can about such figures before trying the test.

QUIZ

Is it true or is it false that

1. If a four-sided framework is braced by two equal diagonal braces, the frame must be a parallelogram?

2. When the opposite beams of a four-sided section of a bridge span are parallel, the section is a parallelogram?

3. Any rectangle is a rhombus?

4. The sum of the interior angles of a parallelogram is four right angles?

5. A trapezium is the framework used to support a trapeze?

6. The sum of two adjacent angles of a parallelogram is 180°?

7. The interior angles of an arrow-headed four-sided figure together make 360°?

8. A baseball diamond is a trapezium?

9. The sum of the exterior angles of a rectangle is four right angles?

10. A four-sided figure whose diagonals bisect each other is a parallelogram?

11. A brace joining the

two legs of a derrick half-way along each leg is parallel to the ground?

12. A four-sided figure with all its sides equal is a parallelogram?

13. Four straight sticks will make one and only one quadrilateral?

14. The sum of the exterior angles of a parallelogram is four right angles?

15. The sum of the opposite angles of a rhombus is 180°?

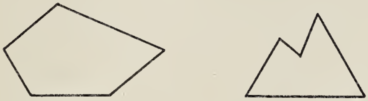
16. When a brick of ice-cream is cut in any direction across the four edges of the brick the section is a parallelogram?

2. Side and angle properties.

The facts which were reviewed in §1 are stated in theorem form at the end of the chapter, being Theorems 17-19. Theorems 20-21 on triangles are derived directly from these. Theorems 20-21 are special cases of two basic theorems in proportion (see chap. VIII).

The facts in Theorems 17-21 should be understood before doing the exercises.

Any polygon not a triangle may be (i) a *convex* figure or (ii) a *re-entrant* figure. This is illustrated in fig. 2 for the *pentagon* (5-sided polygon).



(i) (ii)

Fig. 6.1

You will note that a straight line across (i) cannot intersect the pentagon in more than two points, whereas straight lines can be drawn across (ii) to intersect the sides in four points.

The following example will remind you of the method of stating proofs.*

Example

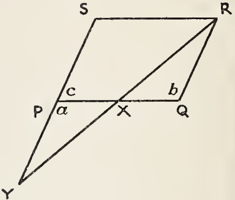


Fig. 6.2

PQRS *is a parallelogram;* **X** *is the mid-point of* **PQ;** **RX** *and* **SP** *are produced to meet at* **Y.** *Prove that* **PY = PS.**

Data ||gram **PQRS**

(1) **PX = XQ.**

To prove that **PY = PS.**

Discussion. If **PY = PS**, then **PY = QR**. Why? You can prove **PY = QR** if Δ**PXY** $\equiv$ Δ**QXR**. This will follow if $\angle a = \angle b$ (A.S.A.). Why?

Construction. No further construction is needed.

Proof

PS || to **QR**,	(Def. ||gram.)
$\therefore \angle a = \angle b$.	(Alt. angles) (Theorem 5(ii))
In Δs **PXY** and **QXR**	
PX = XQ,	(Data (1))
$\angle a = \angle b$,	
$\angle$**PXY** = $\angle$**RXQ**,	(Vert. opp. angles) (Theorem 3)
$\therefore$ Δ**PXY** $\equiv$ Δ**QXR**.	(A.S.A.) (Theorem 8)
Hence **PY = RQ**.	(Corresp. sides)
But **RQ = PS**,	(Opp. sides ||gram) (Theorem 18(i))
$\therefore$ **PY = PS**.	

---

*This method may be applied to the solutions of problems in algebra, chemistry, physics, etc. A study of law court procedure is also of interest in this connection.

EXERCISES. Side and Angle Properties

Quadrilaterals

1. Draw figures to show (1) a convex quadrilateral; (2) a re-entrant quadrilateral; (3) the various kinds of convex quadrilaterals.

2. As a triangle varies in size and shape, there are three variable sides and three variable angles. The sum of the angles is always 180°, however. The specifications for a triangle are as a rule three parts (sides or angles) of the triangle.

(a) Use sticks or metal strips to find the usual number of specifications (sides or angles) for a convex quadrilateral.

(b) Find some simple applications in (a), viz., a garden gate, rectangular, with a diagonal brace.

(c) How is a re-entrant quadrilateral specified?

3. You have seen in a previous exercise (ex. 5, §1, chap. IV) that when the sides of a triangle are produced in order, the sum of the exterior angles of a triangle is 360°.

(a) Find the sum of the exterior angles of a convex quadrilateral.

(b) What is the sum for the convex pentagon (5-sided polygon)?

(c) Is the theorem in (a) true for re-entrant figures?

4. (a) What is the sum of the interior angles of (i) a pentagon; (ii) a hexagon (6-sided polygon); (iii) an n-sided polygon?

(b) Can the figures in (a) be re-entrant?

Constructions

Work from the data with scale, protractor and compasses.

5. On a given straight line construct a rhombus having one of its angles = 60°.

6. Construct a parallelogram having given two sides 8 cm. and 5 cm. and a diagonal 10 cm. Measure the angles.

7. Construct a rectangle given one side 3 in. and a diagonal 5 in. Measure the other side.

8. Construct a parallelogram given the diagonals 3 in. and 4 in. and the angle between them 45°. Measure the sides. (Work without a protractor.)

9. Construct a rhombus given the lengths of the diagonals 10 cm. and 8 cm. Measure the sides.

10. On a base 3 in. long construct a parallelogram of height 1.2 in. with an angle of 55°. Measure the other side.

11. Construct a rhombus each of whose sides is 7.3 cm., the distance between a pair of opposite sides being 5.6 cm. Measure its acute angle.

12. Draw a quadrilateral having one angle equal to 50°, and having each side 2.2 in. long. Measure the longer diagonal.

13. Two vessels start from the same point at the same instant. One steams at 15 knots on a straight course N. 30° E.; the other steams on a straight course S. 50° E. and observes that the former vessel is always due N. of her. Find after what time the vessels will be 30 sea miles apart; also find the speed of the second vessel.

14. From a point O, A bears N. 65° E., B is due E. and C bears S. 40° E. B is known to be in a straight line with A and C and the same distance from each. Find the bearing of A from C.

15. Construct a △ ABC with the median AM = 85 mm., ∠ BAM = 38°, ∠ CAM = 42°.

Parallelograms (Theoretical)

16. ABCD is a parallelogram and O is the mid-point of BD; prove that O is also the mid-point of XY, where XY is any line through O meeting opposite sides in X and Y.

17. The bisectors of two adjacent angles of a parallelogram are at right angles to one another.

18. The bisectors of two opposite angles of a parallelogram are parallel.

19. ABCD is a parallelogram; AB, CD are bisected at X, Y respectively; prove that BXDY is a parallelogram.

20. If the diagonals of a parallelogram are equal, it must be a rectangle.

21. In the diagonal AC of a parallelogram ABCD points P, Q are taken such that $AP = CQ$; prove that BPDQ is a parallelogram.

22. ABCD, ABXY are two parallelograms on the same base and on the same side of it. Prove that CDYX is a parallelogram.

23. E, F, G, H are points in the sides AB, BC, CD, DA respectively of a parallelogram ABCD, such that $AH = CF$ and $AE = CG$; show that EFGH is a parallelogram.

24. Draw a parallelogram ABCD and a rectangle ABPQ such that CD and PQ are in the same line. Prove $\Delta ADQ \equiv \Delta BCP$.

25. Draw two parallelograms ABCD and ABPQ with CD and PQ in the same line. Prove $\Delta ADQ \equiv \Delta BCP$.

26. What figure is formed by joining the mid-points of the sides of a rectangle?

27. ABCD is a square; from A lines are drawn to the mid-points of BC, CD; from C lines are drawn to the mid-points of DA, AB. What figure is enclosed by these lines? Give a proof.

28. In a triangle ABC, BE and CF are drawn to cut the opposite sides in E and F; prove that BE and CF cannot bisect one another.

The Mid-Point Theorems

29. X, Y, Z are the mid-points of the sides BC, CA, AB of a triangle ABC. Prove that AZXY is a parallelogram.

30. The straight lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

31. Show how to construct a triangle, given the three mid-points of its sides. Give a proof.

32. If $AD = \frac{1}{4} AB$ and $AE = \frac{1}{4} AC$, prove that DE is parallel to BC and equal to a quarter of BC.

33. PQR is a triangle right-angled at P; through O, the mid-point of QR, parallels OX, OY are drawn to PQ, PR to meet the sides in X, Y. Prove that $XY = OP = OQ = OR$.

34. If the mid-points of the adjacent sides of a quadrilateral are joined, the figure thus formed is a parallelogram. (Draw a diagonal of the quadrilateral.)

35. The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another.

36. State the converse of the Corollary, Theorem 20. Is it true? Give examples.

3. Area.

The formulas for finding the area of a rectangle, a triangle, and a parallelogram in terms of base and altitude should be recalled.

Theorems 22 and 23 indicate how the proofs of the formulas for the triangle and parallelogram are obtained. The facts suggested by these theorems should be briefly reviewed before attempting the exercises. So also should construction 7 be reviewed.

EXERCISES. Area

1. Find the area of a parallelogram (i) of sides 3.5 in. and 4.5 in. and of angle 70°; (ii) of sides 2 in. and 3 in. and of angle 30°; (iii) of sides 7.85 cm. and 8.75 cm. and of angle 45°.

✓ **2.** Draw a rectangle on base 10 cm. and of altitude 8 cm.; on the same base construct an equivalent parallelogram of angle 60°; measure its longer diagonal.

✓ **3.** Construct a rhombus of side 3.5 in. equal in area to a parallelogram of sides 3.5 in. and 2.75 in. and of angle 70°.

4. A triangle ABC has $AB = 3$ in. and $BC = 4$ in.; trace the changes in its area as $\angle ABC$ increases from 0° to 180°.

5. A parallelogram PQRS has $PQ = 10$ cm. and $QR = 7$ cm.; trace the changes in its area as $\angle PQR$ increases from 0° to 180°.

6. Find, to three significant figures, the area of each triangle, taking the mean of the results obtained with each of the three altitudes of the triangle.

(i) sides 3, 4, 4.5 in.;
(ii) sides 6, 8, 9 cm.;
(iii) sides 3, 4, 5 in.;
(iv) sides 2, 3, 4.5 in.;
(v) sides 3, 4 in., included $\angle 120^\circ$;
(vi) $BC = 7.2$ cm., $\angle B = 20^\circ$, $\angle C = 40^\circ$.

7. The section of a trough is triangular, the sides being 15 in. and the width 12 in. Find the sectional area.

8. Find the surface (i.e., the sum of the areas of all the faces):

(i) of a tetrahedron (three-sided pyramid) each of whose edges is 4 inches;
(ii) of a square pyramid whose base edges are each 2 inches and whose slant edges are each 2.5 inches;
(iii) of a regular 3-sided prism whose base is an equilateral triangle of side 2 inches and whose height is 3.5 inches.

✓ **9.** A prism stands on a triangular base of sides 5 in., 12 in., 13 in., and is 1 ft. high. Find the whole surface area.

10. In ex. 9 a plane through the longest side of the base cuts the opposite parallel edge at its mid-point. Find the lengths of sides of the triangular section and thence the area of the section.

11. Find a formula for the area of any trapezium.

✓**12.** Draw a trapezium **ABCD** with **AB** parallel to **CD**, given $AB = 4$ in., $AD = 2.8$ in., $DC = 2$ in., $\angle A = 70^\circ$. Find its area.

13. A symmetrical railway cutting has a cross-section 20 ft. wide at bottom, 80 ft. at top, and sloping sides 45 ft. long. Find, by drawing and measurement, the area of the section.

14. Fig. 6.3 represents an oval boiler plate. To find its area the central line is divided into 8 equal parts

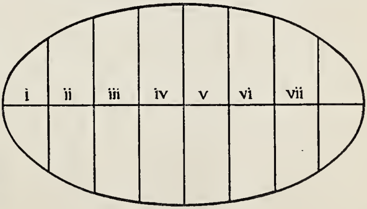


Fig. 6.3

each of length 3 in. The cross-lines numbered i-vii are of lengths 8 in., 11.5 in., 13 in., 14 in., 13 in., 11.5 in., 8 in. respectively. By regarding each portion formed by these cross-lines as a trapezium, find the area of the plate.

15. Find the area of the figure whose vertices have co-ordinates (1, 2), (1, 10), (6, 13), (6, 2).

16. Find the area of the triangle (1, 2), (2, 5), (3, 1) by adding two trapezia and subtracting one. (Draw perpendiculars to the x – axis.)

17. In calculating the area of a piece of land, fig. 4, a surveyor's book had the following entries:

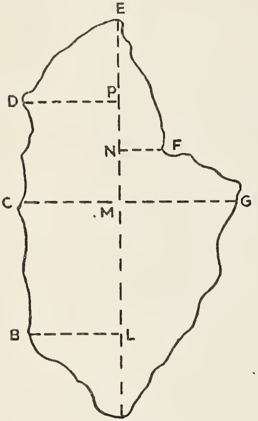


Fig. 6.4

Yards

	To E	
	600	
240	460	
	360	50
240	300	120
200	100	
From	A	go North

This record is to be read upwards. In the middle column are set down the distances from A of the different points on the baseline; on the right and left are set down the offsets as they occur; e.g., L is 100 yards north of A, and B is 200 yards to the left of L; and so on.

Draw a plan of the field represented in fig. 4 from the measurements given. Make the plan on graph paper and state clearly the scale used. Calculate the area of the field in sq. yd., ignoring the small boundary irregularities.

18. Draw a plan (freehand) and find the area of the fields in the following surveys:

Yards

	To **D**	
	400	
	340	50
70	300	
90	200	
30	100	50
From	**A**	go north

Yards

	To **B**	
	500	
	470	100
70	400	140
	250	150
100	100	
From	**A**	go north

19. Reduce the quadrilateral ABCD to a triangle and find its area, when

(i) $DA = 1$ in., $\angle A = 100^\circ$, $AB = 2.3$ in., $\angle B = 64^\circ$, $BC = 1.5$ in.;

(ii) $AB = 5.7$ cm., $BC = 5.2$ cm., $CD = 1.7$ cm., $DA = 3.9$ cm., $\angle A = 76^\circ$.

20. Reduce a pentagon ABCDE to a triangle with base along AB and vertex at D.

21. Find the area of a pentagon ABCDE, given $AB = 6.5$ cm., $BC = 2.4$ cm., $CD = DE = 4$ cm., $EA = 2.5$ cm., $\angle A = 80^\circ$, $\angle B = 133^\circ$. (Reduce the pentagon to a triangle.)

22. A rectangular sheet of cardboard 30 cm. by 24 cm. weighs 52 gm. An irregular figure is traced on to it, cut out, and found to weigh 21.4 gm. What is the area of the figure? (A method of area finding by weighing.)

23. Use your local sectional sheet map to find areas of land or water regions suggested by the map. A suitable unit of area is the square mile (section). The regions may be taken off the map by tracing on thin paper. If a sectional map is not available use the air navigation map at the end of the book.

4. The Theorem of Pythagoras.

You will possibly have proved this famous theorem in an earlier course.

THEOREM 24. **In a right triangle the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.**

An outline of the proof of the theorem is given in chap. X.

Particular cases of the relation were known in Egypt, Chaldea, and China long before the time of Pythagoras (c. 550 B.C.). There is some doubt as to the origin of the general theorem; it may have been proved by the Pythagoreans with the use of similar triangles. The proof outlined in chap. X is probably Euclid's own (c. 300 B.C.).

The converse of Theorem 24, stated at the end of this chapter as Theorem 25 is also true. The converse theorem leads to a method of constructing a right angle which was perhaps used by the ancient Egyptians as early as 2000 B.C., in constructing the ground plan of temples and pyramids. It is still the best method for marking out the right angles at the corners of a lawn tennis court and in much field work.

Suppose that the base-line has been marked. On a piece of rope mark off three lengths of 3, 4, 5 yards. Put in a peg at one end of the base-line, another peg on the base-line 4 yards from the corner. Stretch the rope round these two pegs, so that the ends of the 4-yard length coincide with the two pegs. Now bring the two ends of the rope together so that a triangle is formed with sides 3, 4, 5 yards. This will give a right angle at the corner, and the side line can be marked out.

EXERCISES. Right Triangles*

1. Are the triangles right-angled whose sides are: (i) 8, 17, 15; (ii) 12, 36, 34; (iii) 25.5, 25.7, 3.2; (iv) $4n$, $4n^2-1$, $4n^2+1$; (v) m^2+n^2, m^2-n^2, $2mn$; (vi) a, b, $a+b$?

2. ABC is a triangle and BD is perpendicular to AC. If $\mathsf{AD} =$ 8 in., $\mathsf{DC} = 4.5$ in., $\mathsf{BD} = 6$ in., show that $\angle \mathsf{ABC}$ is a right angle.

---

* See chap. II, §3 for preliminary exercises.

3. Namao is 11.6 mi. from Fort Saskatchewan by map measurement. Ellerslie is 20 mi. due south of Namao, and is 23.1 mi. from Fort Saskatchewan. Is Fort Saskatchewan due east of Namao?

Three Dimensions

4. Find the slant side of a cone of (i) height 5 in., base-radius 3 in.; (ii) height 25 mm., base-diameter 56 mm.; (iii) height h ft., base-diameter $2r$ feet.

5. Find the height of a cone of (i) slant side 20 in., base-radius 6 in.; (ii) slant side 6.2 m., base-diameter 10 m.; (iii) slant side 1 ft., base-radius r ft.

6. Find the base-radius of a cone of (i) slant side 11.3 cm., height 57 mm.; (ii) slant side 1 in., height h in.

7. A pyramid of height 5 in. stands on a square base of side 4 in. Find the length of a sloping edge of the pyramid.

8. At two points, one due E. and the other due S. of a tower 100 ft. high, the tower subtends an angle of 45°. What is the distance between the two points?

9. **ABC** is the base of a right prism, **AB** = 9 in., **BC** = 8 in., **CA** = 12 in. **P** is on the edge through **A** perpendicular to the base and **AP** = 5 in. Calculate the sides of the triangle **PBC**.

10. In ex. 9, **Q** is on the perpendicular edge at **B** and **BQ** = 4 in., **R** on that at **C** and **CR** = 6 in. Calculate the sides of the triangular section **PQR**.

11. The street poles for a trolley-bus wire are 30 ft. high and are braced 5 ft. from the top to a point 15 ft. from the base of the pole. How long is the brace wire?

12. In fig. 5, which is a portion of a steel frame, **OX** = **OY** = 10 ft., **XR** = **RY**, **OZ** = 15 ft. and **OX**, **OY**, **OZ** are at right angles to one another. Show that **OR** is perpendicular to **XY**. **OR** is also perpendicular to **OZ**. Why? Find the length of **RZ**.

13. Invent some examples of the use of the Pythagorean theorem.

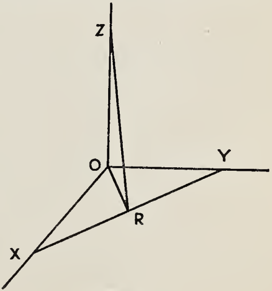


Fig. 6.5

III. Side and Angle Properties of Quadrilaterals

Terms: Convex and re-entrant quadrilaterals.
Opposite sides and angles; diagonals.
The classification of quadrilaterals:—trapezium, parallelogram, rhombus, rectangle, square.

THEOREM 17. **The sum of the angles of a quadrilateral is equal to four right angles.**

THEOREM 18. **If a quadrilateral is a parallelogram, then (i) the opposite sides are equal; (ii) the opposite angles are equal; (iii) each diagonal divides the parallelogram into two congruent triangles; (iv) the diagonals bisect each other.**

THEOREM 19. (CONVERSE OF THEOREM 18.) **A quadrilateral is a parallelogram:**

(i) **when both pairs of opposite sides are equal;**
or (ii) **when both pairs of opposite angles are equal;**
or (iii) **when one pair of opposite sides are equal and parallel;**
or (iv) **when its diagonals bisect each other.**

IV. Triangles. Mid-point Theorems

THEOREM 20. **The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.**

COROLLARY. **Given three parallel straight lines and two transversals of these lines. If one transversal is bisected by the parallels, so also is the other transversal bisected.**

THEOREM 21. (CONVERSE OF THEOREM 20.) **The straight line joining the middle points of two sides of a triangle is parallel to the third side and equal to half of it.**

V. Area

Terms: Equi-areal or equivalent figures; base and altitude of a triangle; of a parallelogram.

Theorem 22. **Parallelograms on the same base and with the same altitude are equi-areal.**

Corollary 1. **The area of a parallelogram is measured by the product of base and altitude.**

Corollary 2. **Triangles on the same base and with the same altitude are equi-areal.**

Corollary 3. **The area of a triangle is measured by one-half the product of base and altitude.**

Theorem 23. (Converse of Corollary 2, Theorem 22.) **Equi-areal triangles on the same base have the same altitude. If the triangles are on the same side of the base, they are between the same parallels.**

Construction 7

7. *To construct a triangle equal in area to a given quadrilateral.*

The area of a rectilinear figure by (1) decomposition into triangles; (2) the construction of a triangle equal in area to the given figure.

The area of a curvilinear figure by scale drawing on co-ordinate paper and counting squares.

VI. The Side Property of a Right Triangle

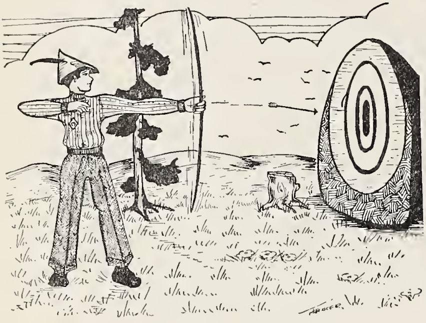
Theorem 24. **In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.**

Theorem 25. (Converse of Theorem 24.) **If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.**

CHAPTER VII

LOCI

1. The locus of a point.



Try to imagine a point moving freely about in space. Actually such an idea is hard to realize. A fluttering leaf, a bit of flying waste paper, these give an appearance of random motion; in reality they are moving under the influence of wind, temperature, pull of gravity, etc.

Whenever a point moves under certain conditions (restrictions), the path traced by the point is called its *locus* (locus = place; pl. *loci*).

EXERCISES. The Locus of a Point

1. What is the locus described by:

   (i) A falling stone?

   (ii) Two corners of a set-square as it lies flat on a desk and rotates about the third corner?

   (iii) A man as he walks N.E. from a fixed point?

   (iv) The centre of a nickel as it rolls around the edge of a fixed quarter?

(v) The handle of a door as the door opens?

(vi) The centre of one end of a boat roller as the boat moves?

2. What is the locus of a point as it moves in a plane so that it is always

(i) 2 in. from a given line?

(ii) 2 in. from a given point in the plane?

(iii) At the same distance from a fixed point **A** as from another fixed point **B**? Sketch to show a few positions of the moving point.

(iv) At equal distance from two intersecting lines? Sketch as in (iii)?

(v) Equidistant from two parallel lines? Sketch as in (iii).

3. What is the locus in space of the centre of a sphere rolling in any way—

(a) on a table?

(b) inside a hollow cylinder?

(c) inside a hollow sphere?

(d) inside a hollow cone?

4. What is the locus in space of a point which moves so that it is always

(i) 3 ft. from a given plane?

(ii) 3 ft. from a given line?

(iii) 3 ft. from a given point?

(iv) 3 ft. from the surface of a given sphere?

(v) equidistant from two parallel lines?

(vi) equidistant from two intersecting planes?

(vii) equidistant from two points?

(viii) equidistant from two intersecting lines?

5. A point **O** is 2 in. from the centre of a circle of radius 2.5 in. Plot the locus of the mid-point of **OP**, when **P** moves round the circumference of the circle.

6. A rod **OP** is freely hinged at **O**; what is the locus of **P**? At **P** another rod **PQ** is freely hinged and always hangs vertically; what is the locus of **Q**? Loci may be considered in the plane and in space.

7. Draw **ABCD** a parallelogram. Now regard the figure as a framework freely jointed at **A**, **B**, **C**, **D**. If **AB** is fixed, what are the loci (i) of **C**, (ii) of **D**? Loci may be considered in the plane and in space.

8. In the preceding question, if the diagonals intersect in **O**, what is the locus of **O**?

9. A packing case is rolled about one edge; sketch the locus of a point on the opposite edge.

10. You will likely have been finding loci of your own, viz.,

(a) the path of a baseball when thrown into the air (1) without wind resistance; (2) with strong wind resistance;

(b) the path of the knot of a ribbon tied to the spoke of a bicycle wheel when the bicycle is wheeled along a level road.

Find other examples.

2. Freedom and authority.

The locus idea illustrates the exercise of freedom under constraint, an idea which is basic to the development of democracy.

"To find the right adjustment between liberty and authority has always been the most difficult of political problems."*

We merely draw attention to this problem to say that the idea of freedom under constraint is also far-reaching in applied mathematics.

Equations (other than "identities") are restrictions on the "freedom" of variables. $2x+y=3$ links x and y together. You may think of x as "free," with y dependent on x. Thus, if $x=5$, y must be -7, etc.

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*"Psychology and Modern Problems," essay by Muir on *Liberty, Authority. Democracy* (Univ. of London Press, 1935), p. 111.

You should note that "inequalities" are also restrictive; e.g., $\mathbf{G} \leqq 5000$ represents the gallonage of a locomotive tender. The capacity of the tender is 5000 gal. **G** is restricted; it cannot exceed 5000.

EXERCISES. Freedom and Authority

1. An eavestrough of rectangular cross-section is made from a long metal strip 9 in. wide. The depth of the trough is x in. and the width is y in.

(a) What are the restrictions on the size of x? On the size of y?

(b) How is y related to x?

2. A tank has a capacity of 1000 gal. and contains 300 gal. to begin with. Water flows into the tank at a rate of 25 gal. per hour until full. y is the number of gallons in the tank after x hours.

(a) How many gallons are in the tank at any time?

(b) "At any time." What are the restrictions on x? on y?

3. Here are some examples of restrictive statements. Find other examples.

(a) Traffic instruction: "One hour parking in this block."

(b) Stock quotation, Oct. 1, 1939: "A.B. Steels, $35\frac{1}{2}$ low, $52\frac{1}{2}$ high."

4. A recent judgment of the Supreme Court of Canada contains the following statement: "Democracy cannot be maintained without its foundation: free public opinion and free discussion throughout the nation of all matters affecting the State within the limits set by the criminal code and the common law."

According to the above quotation:

- (i) What matters affecting the State should you be free to discuss publicly?
- (ii) When should you be restrained from such discussion?
- (iii) What is the law on these matters?

3. The intersection of loci.

In §1, ex. 2 you found the locus of a point (a) equidistant from two fixed points; (b) equidistant from two fixed intersecting straight lines. These loci are stated in theorem form at the end of the chapter.

The intersection of a pair of loci (which are usually straight lines or curves) will determine a point or points.

EXERCISES

Draw good figures, and give explanations in locus terms.

1. Draw an unlimited straight line and mark a point O 2 inches from the line. Find a point (or points) 3 inches from O and 3 inches from the line. (What is the locus of points 3 inches from O? What is the locus of points 3 inches from the line? Draw these loci.) Measure the distance between the two points found.

2. In ex. 1 find as many points as you can distant 1 inch from both point and line.

3. If A, B are two points 7 cm. apart, show how to find a point (or points) distant 9 cm. from A and 5 cm. from B.

4. Make an angle of 40°; on one of the arms mark a point A 6 cm. from the vertex of the angle. Find a point (or points) equidistant from the arms of the angle, and 5 cm. from A. Measure the distance between the two points found.

5. Draw two unlimited straight lines. Find all points 1 inch from one line and 1.5 inches from the other.

6. Draw a triangle ABC; find a point O which is equidistant from B, C; and also equidistant from C, A. Test by drawing a circle with centre O to pass through A, B, C.

7. Draw a triangle ABC. Inside the triangle find a point P which is equidistant from AB and BC; and also equidistant from BC and CA. From P draw perpendiculars to the three sides; with P as centre and one of the perpendiculars as radius draw a circle.

8. Draw a triangle **ABC**; find a point **O** which is equidistant from **B** and **C** and also equidistant from **AB** and **AC**.

9. Two straight railway tracks converge at an angle of 40°. A shed is to be erected 200 yards from the junction and equidistant from each track. Draw a figure showing its position and find its distance from each track.

10. A harbour is formed by two breakwaters **AB** and **BC**. **AB**, 600 yds. long, is perpendicular to the shore, and **BC** makes an angle of 70° with **AB**. A ship is anchored 150 yds. from **BC** and 250 yds. from **AB**. How far is it from the shore?

11. A vessel is steering a course parallel to a straight shore and $\frac{1}{2}$ mile distant from it. Over what distance would it be in danger from torpedoes fired from a point on shore if the effective range of a torpedo is 1 mile?

12. A weight hanging by a string $3\frac{1}{2}$ ft. long from a point in a wall is pulled until it is 2 ft. away from the wall. What angle does the string now make with the wall? Use a protractor.

13. The position of hidden treasure in a triangular field is described as being equidistant from two sides and also equidistant from two corners. Could this statement be true?

14. An electric light hangs 2 ft. from the ceiling and 3 ft. from a wall. What possible positions may it have? If it is 5 ft. from one end of the room, where may it be? (Note the use of three loci here.)

15. (a) Three fixed points which are not on a straight line fix a plane. (i) Find the point or points in this plane equidistant from the three fixed points; (ii) how many conditions are present? (iii) are they all independent of each other?

(b) What is the locus in space of a point equidistant from the three fixed points in (a)?

16. An aeroplane reports that it is flying north-west towards an airport and is descending at an angle of 30°. Draw its locus. A moment later it reports an altitude of 2000 ft. Locate the plane with respect to the airport and find its ground distance from the port.

4. The use of co-ordinates.

From your work in Algebra you are familiar with the use of co-ordinates.

A point is represented by an ordered pair of numbers (x, y). These numbers are called co-ordinates, the first being the x-co-ordinate and the second the y-co-ordinate of the point.

When a point is subject to a restriction on its freedom in the plane, x and y will be restricted, often by an equality such as $y=3x+2$. In this instance the locus of the point with co-ordinates (x, y) is a straight line whose equation is $y=3x+2$.

EXERCISES. The Use of Co-ordinates

1. In each of the following the locus of a point (x, y) whose co-ordinates satisfy the given condition, is a straight line. Draw the line.

(a) $x=2$; (b) $y=3$; (c) $x=-1$;
(d) $y=-4$; (e) $x=0$; (f) $y=0$.

2. As in ex. 1.

(a) $y=x$; (b) $y=x+2$;
(c) $y=2x+1$; (d) $4x-2y+3=0$.

3. In each of the following the locus of a point (x, y) whose co-ordinates satisfy the given condition, is a curve. Draw the curve.

(a) $y=x^2$; (b) $y^2=4x$

5. Other locus examples.

The locus of a (variable) point which is subject to certain given conditions consists of all the possible positions of the point. The locus is usually a curve, but not always.

EXAMPLES

1. *A point is* (i) 3 *in. from* A; (ii) 4 *in. from* B, *where* AB $=6$ *in. What is the locus of the point?*

(i) *All* the points 3 in. from A lie on a circle, centre A, radius 3 in.;

(ii) *All* the points 4 in. from A lie on a circle, centre A, radius 4 in.

Hence there are two and only two points satisfying conditions (i) and (ii), namely, the two points of intersection of the above circles.

2. *The co-ordinates* (x, y) *of a point satisfy the equation* $y^2 = (x+1)^2 (x-2)$. *What is the locus?*

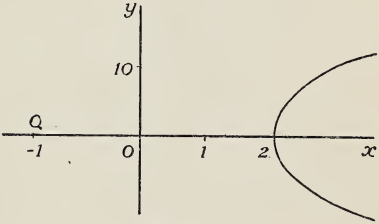


Fig. 7.1

All the points (and no others) which satisfy the given equation must be found. The graph is shown in fig. 1. It consists of a curve and a single point Q $(-1, 0)$. The graph is hardly to be called a path.

3. *What is the locus of points for which* $x<1$*?*

Such points will lie in the plane to the left of AB. The locus is that portion of the plane to the left of AB. Points on AB are not "on" the locus. Points such as Q$(-3, 5)$, R$(0.99, 2)$ belong to the locus. Why?

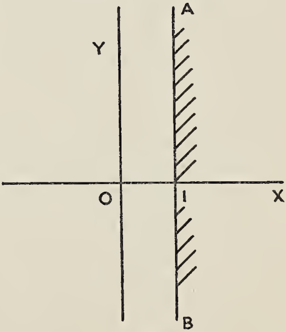


Fig. 7.2

EXERCISES. Other Locus Examples

1. Find *all* the points in the plane (and no others) whose co-ordinates (x, y) satisfy the given conditions:

(a) $x^2=1, \quad y=0;$
(b) $x=4, \quad y^2=25;$
(c) $x^2=4, \quad y^2=1;$
(d) $y=x, \quad y=2x-3;$
(e) $y=x^2, \quad y=4x+5.$

2. Find *all* the points in the plane (and no others) whose co-ordinates (x, y) satisfy the given condition:

(a) $x>0;$
(b) $y>1;$
(c) $1<x<3;$
(d) $-1<y<1;$
(e) $x<-2.$

These loci are plane *regions*, not curves. Such regions are of fundamental importance in the study of curves and their equations.

VII. Simple Loci in the Plane

THEOREM 26. **The locus of points which are equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.**

THEOREM 27. **The locus of points which are equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.**

CHAPTER VIII

SIMILAR FIGURES

1. Ratio and proportion.

One of the commonest ways of comparison is by division; thus we say that the length a in. is twice length b in.; hence $\frac{a}{b} = 2$. The quotient $\frac{a}{b}$ is spoken of as the *ratio of a to b*, often written as $a : b$.

The comparison of figures may reveal proportionality. A snapshot and its enlargement are in proportion.

Let their breadths be b in. and b' in., and let their lengths be l in. and l' in. Then the ratio $b : b'$ of the breadths is equal to the ratio $l : l'$ of the corresponding lengths, that is

$$b : b' = l : l'.$$

The equality $b : b' = l : l'$ is called a *proportion* and is read as follows:

b is to b' as l is to l'.

The proportion may of course be written as

$$\frac{b}{b'} = \frac{l}{l'}$$

to be read as above.

In any proportion $a : b = c : d$ we say that *a and c are proportional to b and d.* Hence, above, we should say that the breadth and length of the snapshot are proportional to the breadth and length of the enlargement.

Proportion may exist for more than two pairs of numbers. The diagonal and breadth of the snapshot are proportional to the diagonal and breadth of the enlargement: $\frac{d}{d'} = \frac{b}{b'}\cdot$

The proportion for breadths, lengths and diagonals is written as: $\frac{b}{b'}=\frac{l}{l'}=\frac{d}{d'}$.

This is a convenient way of writing the separate proportions:

$$\frac{b}{b'}=\frac{l}{l'},\quad \frac{b}{b'}=\frac{d}{d'},\quad \frac{l}{l'}=\frac{d}{d'},$$

of which only two are independent equations.

EXERCISES. Ratio and Proportion

1. Write descriptive sentences for each ratio as in the sample.

(a) (*Sample*). *Area (sq. in.).* $\frac{A_1}{A_2}=0.25.$

The areas A_1 sq. in. and A_2 sq. in. are in the ratio 1 *to* 4 OR *the area A_1 sq. in. is one-fourth the area A_2 sq. in.*

(b) Volume (cu. ft.). $\frac{V_1}{V_2}=1.5;$

(c) Length (miles). $\frac{x}{y}=0.75;$

(d) Density (gm./cu. cm.). $\frac{D}{D'}=\frac{6}{5}$.

2. (a) John said, "If, as in the sample ex. 1, the areas are in the ratio 1 to 4, I shall write this ratio as a ratio of areas $\frac{A_1\ sq.\ in.}{A_2\ sq.\ in.}=0.25$." Write the remaining parts of ex. 1 in this manner.

(b) Dan asked John, "If two areas are 120 sq. yd. and 800 sq. ft., what is their ratio?" John replied, "The ratio of the areas is $\frac{120\text{ sq. yd.}}{800\text{ sq. ft.}}=\frac{9\times 120\text{ sq. ft.}}{800\text{ sq. ft.}}=13.5$."

Invent statements like this for the remaining parts of ex. 1.

3. Draw a line 5 in. long. Calculate the length of $\frac{3}{8}$ of the line and cut off this length from the line. In what ratio is the line now divided?

4. If you want to divide a line in the ratio 3 : 4, what portion of the line must you cut off?

5. Draw line segments of various sizes and by inspection divide these in ratios of 5:3, 1:4, etc.

✓**6.** A line 5 in. long is divided in the ratio of 3 : 4. Calculate the lengths of the parts.

7. A line is divided in the ratio of $a:b$. What expressions give the lengths of the two parts?

✓**8.** A line 12 cm. long is divided at C in the ratio of 2 : 3, and at C′ in the ratio of 3:5. What is the length CC′?

9. The sides of a triangle are 6, 8, 7 in. long. The side 7 in. is divided in the ratio of the other two sides. What are the lengths of the parts?

✓**10.** Two circles have radii 10 cm., 15 cm., and the distance between their centres is 30 cm. How far from the centre of the larger is a point which divides the distance between their centres in the ratio of their radii, the larger segment being measured from the centre of the larger circle?

11. A line 27 ft. long is divided into 3 parts in the ratio of 2:3:4. What are the lengths of the segments?

✓**12.** Write the variables (letter and unit) and the proportion indicated, as in the sample.

(a) (*Sample.*) *The base and altitude of a given triangle are proportional to the base and altitude of any enlargement of the given triangle.*

The bases are b_1 *cm. and* b_2 *cm. and the altitudes are* h_1 *cm. and* h_2 *cm. The proportion is* $\frac{b_1}{b_2} = \frac{h_1}{h_2}$.

(b) Given a triangle and its enlargement.
  (i) The bases are proportional to the altitudes;
  (ii) The areas are proportional to the squares of the bases;
  (iii) The areas are proportional to the products of bases and altitudes.

(c) For a certain period of their growth the weights of plaice are proportional to the cubes of their lengths.

13. (a) If $\frac{a}{b} = \frac{c}{d}$ prove that $\frac{a}{c} = \frac{b}{d}$. Verify by taking numerical values for a, b, c, d.

(b) State what you have proved in words; make use of the word *proportional.*

14. Find x if:

(i) $\frac{3}{5} = \frac{x}{20}$, (ii) 6 in : 13 in. $=$ 5 yd. : x yd.,

(iii) 30 cm. : x in. $=$ 25 cm. : 45 in.

15. Are the following true?

(i) 13 cm. : 0.7 in. $=$ 32.5 cm. : 1.75 in.,

(ii) $\frac{315}{276} = \frac{4.2}{3.68}$,

(iii) $\frac{41}{90} = \frac{42}{91}$,

(iv) $\frac{a}{b} = \frac{a+1}{b+1}$,

(v) $\frac{a}{b} = \frac{ma}{mb}$, $m \neq 0$.

16. Complete the statements, using equalities.

(a) If a, c, e are proportional to b, d, f then

. . . .

(b) If X, Y, Z are proportional to P, Q, R, then
....

(c) Two boxes have exactly the same shape. Their lengths, breadths, depths, face diagonals, and main diagonals are all in proportion. Hence, using letters for these dimensions, ...

2. Proportional division.

Any study of similar figures requires the use of certain theorems about the proportional division of straight line segments.

THEOREM 28. **If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.**

COROLLARY. **If a straight line is drawn parallel to one side of a triangle, the two triangles so formed have their corresponding sides in proportion and their corresponding angles equal.**

THEOREM 29. (CONVERSE OF THEOREM 28.) **If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.**

Hints for the proofs of these propositions are given in chap. X.

3. Similar figures.

You are familiar with many instances of similar figures: snapshot and enlargement, an aeroplane and its scale model, etc.

We say that figures are similar when they have the same shape. This is a poor working definition because of the vagueness of the word *shape*.

You should not proceed to the next paragraph until you have made (a) a list of examples of similar figures; (b) attempts at framing a working definition of similar figures; (c) tests of your definition in (b).

4. The definition of similar figures.

Fig. 1 shows a lake, drawn on graph paper to two different scales, one being half the other.

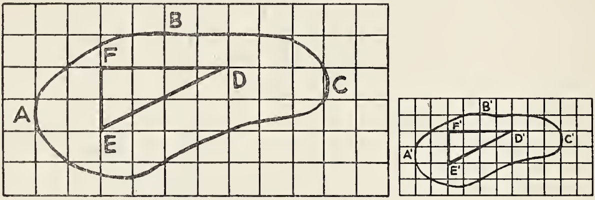


Fig. 8.1

Corresponding points in the figures are easily determined by use of the lines of the grids.* Thus the points A . . . E and A′ . . . E′ correspond. Corresponding lengths are proportional, e.g.,

$$\frac{AB}{A'B'} = \ldots = \frac{DE}{D'E'}$$

Moreover, corresponding angles are equal, viz., $\angle BAC = \angle B'A'C'$, etc.

For a working definition of similar figures it is natural perhaps to use the proportionality of length:

Definition. *Similar figures. Two figures are said to be similar if* (i) *they may be put into a one-to-one correspondence of their points;* (ii) *corresponding lengths are proportional throughout the two figures, i.e., the ratio of the lengths of corresponding line segments in the two figures is the same no matter which pair of segments is used.*

Do you think that two triangles with the sides of the one double the corresponding sides of the other similar? What of two such parallelograms?

**Grid*—A useful word to describe the network of lines on graph paper.

The above definition applies to either plane or three dimensional figures—to drawings of biological sections or to plaster models. Note that corresponding points may be taken anywhere in the figures, not merely on their boundaries.

You may ask, "How are corresponding segments identified?" The answer is in fig. 1,—"Use the grids."

Two corresponding segments DE and D′E′ may always be referred to the grids by means of right triangles, such as DEF and D′E′F′. The sides of these triangles are in proportion.*

EXERCISES. The Definition of Similar Figures

1. Make a list of similar figures, stressing practical processes, e.g., the use of a moving picture projector.

2. What kind of figures are obtained from the above definition when the ratio of corresponding lengths is one?

3. (a) Use the air navigation map to copy the course Lethbridge—Raymond—Cardston—Pincher Creek—Macleod—Lethbridge to scale as follows:

- (i) Label the points L, R, C, P, M on the map;
- (ii) Draw two lines at right angles on your sheet to represent the parallel of latitude 49°30′ and the meridian 113°30′. Let their intersection be O and label the parallel Ox and the meridian Oy;
- (iii) Measure the x- and y- distances in inches for each of the points on the map and complete the table:

	L	R	C	P	M
x					
y:					

---

*We mean by *sides*, the three sides, including the hypotenuse. Note also that if the sides of two right triangles are proportional, the hypotenuses will correspond to one another, since the hypotenuse is the greatest side of a right triangle.

(iv) Plot these values on your sheet (inch co-ordinate paper may be used, if available). Show the scale 8 mi. to 1 in. on your sheet.

(b) Use the results of (a)(iii) to make a new map of the course, scale 10 mi. to 1 in., in the same manner. You should prepare a table like that in (iii). Label this course **L′R′C′P′M′** and show the scale 10 mi. to 1 in. on the sheet.

(c) Use a scale graduated in inches to verify that the ratio of any pair of corresponding distances in the two figures is constant, viz., **CP : CL = C′P′ : C′L′**. What is this ratio?

(d) Verify by measurement that corresponding angles are equal, viz., ∠**MPL** = ∠**M′P′L′**.

(e) Locate any three points **D**, **E**, **F** interior to the course (first sheet) and the corresponding points **D′E′F′** (second sheet) showing their x- and y- co-ordinates as in (a)(iii).

(f) Verify that the ratio of the corresponding sides of the two triangles in (e) is constant. What is this ratio? Also verify that corresponding angles in the triangles are equal.

4. The figures are right triangles. Use the Pythagorean theorem to find

(a) r' in terms of r if:
   (i) $x=5$, $y=10$;
       $x'=4$, $y'=8$;
   (ii) $x'=3x$, $y'=3y$;
   (iii) $4x'=x$, $4y'=y$;

(b) y' in terms of y if:
   (i) $x'=2x$, $r'=2r$,
   (ii) $3x'=x$, $3r'=r$.

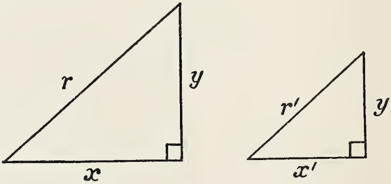


Fig. 8.2

5. Summer, 1939. The negotiations of Britain, France and Russia were reported to have deadlocked over the question of what constitutes aggression. List some of the possible meanings for the vague phrase "aggressor nation".

6. George was asked to say what speed meant. He replied, "Speed is the rate of change of distance with respect to time." What are the vague words in his definition?

7. Try to frame a satisfactory definition for *locus*. Does your definition cover the examples of §5, chap. VII?

5. Similar triangles.

Fig. 3 shows two triangles, the smaller being the larger drawn to half scale.

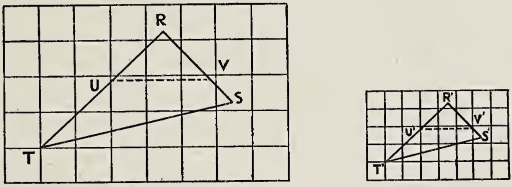


Fig. 8.3

The triangles are in one-to-one point correspondence and all corresponding lengths are in the same proportion, e.g., $UV = 2U'V'$, $TV = 2T'V'$, etc.

In particular, corresponding sides of the triangles are proportional : $\dfrac{RS}{R'S'} = \dfrac{ST}{S'T'} = \dfrac{TR}{T'R'}$.

THEOREM 30. **If two triangles are similar, corresponding sides are proportional.**

Does the proportionality of corresponding sides guarantee similarity? Since a triangle is fixed when its sides are fixed, it is reasonable to expect that a pair of triangles will be similar when corresponding sides are proportional. This we shall assume, without proof.

‡THEOREM 31. (CONVERSE OF THEOREM 30.) **If two triangles have corresponding sides proportional they are similar.**

COROLLARY. **If two right triangles have two pairs of corresponding sides proportional, the ratio of the third pair of sides is equal to that of the other pair, and the triangles are similar.**

Fig. 3 indicates that two similar triangles have corresponding angles equal. The proof of this theorem is outlined in chap. X. So for the succeeding theorems.

THEOREM 32. **If two triangles are similar, corresponding vertex angles are equal, i.e., the triangles are equiangular.**

It is also true that the equality of corresponding angles guarantees similarity.

THEOREM 33. (CONVERSE OF THEOREM 32.) **If two triangles are equiangular, the triangles are similar.**

COROLLARY. **Two right triangles with one pair of corresponding acute angles equal are similar.**

Theorems 30 and 32 give what are often called *necessary conditions* for similar triangles. If two triangles are similar then it *must* be true that (1) corresponding sides are proportional; (2) the triangles are equiangular.

Clubs usually lay down conditions for membership. In order to be a member of a certain hockey team you must: (1) be over 14 years of age; (2) pass a specified physical examination; (3) promise to obey certain training rules, etc. All these conditions are *necessary conditions* for team membership.

Theorems 31 and 33 give *sufficient conditions* for similar triangles. They provide "sufficient grounds" for similarity; they guarantee similarity. Another such guarantee is provided in the following theorem:

THEOREM 34. **If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar.**

The table below contrasts the congruence and similarity properties of triangles. Note that these are sufficient conditions for congruence and similarity.

Triangles

Congruence	*Similarity*
Two triangles are congruent with:	Two triangles are similar with:
A. One side and two angles equal. (Th. 9.)	A. Two angles equal. (Th. 33.)
B. Two sides and included angle equal. (Th. 8.)	B. Two sides proportional and included angle equal. (Th. 31.)
C. Three sides equal. (Th. 10.)	C. Three sides proportional. (Th. 31.)

The use of brief inexact phrases such as "two sides proportional" serves to set out the contrast more sharply.

You should state these conditions accurately and in detail; check your statements against those of the theorems.

Right Triangles

Congruence	*Similarity*
Two right triangles are congruent with:	Two right triangles are similar with:
A. Two sides equal. (Cor. Th. 8 and Th. 13.)	A. Two sides proportional. (Cor. Th. 31.)
B. One side and one acute angle equal. (Cor. Th. 9.)	B. One acute angle equal. (Cor. Th. 33.)

State these conditions fully as for the preceding summary.

EXERCISES. Similar Triangles

Remember that if two triangles are similar they (1) *have their corresponding sides in proportion;* (2) *are equiangular. It is important to look out for corresponding angles and corresponding sides in triangles which are recognized to be similar.*

You may wish to use the symbol ||| *to indicate similar figures.*

A. SIMILAR RIGHT TRIANGLES

1. Draw a triangle sides 6, 8, 10 cm. Construct an equiangular triangle having its shortest side 4.8 cm. Measure the remaining sides and verify by calculation.

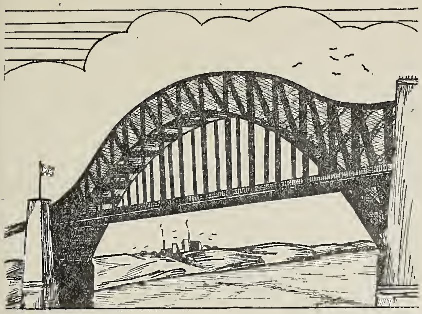
2. PQR is a triangle with QR = 12 in., RP = 5 in., PQ = 13 in. Draw an equiangular triangle UVW having VW corresponding to QR with VW = 8.5 in. Calculate UV, UW and check by measurement.

3. A ladder 30 ft. long is held with one end on the ground resting on a wall 8 ft. high. The length along the ladder from the ground to the top of the wall is 10 ft. What height is the end of the ladder above the ground?

4. A spot light is 2 ft. in front of the middle of a board 30 in. wide suspended vertically. What is the width of the shadow of the board 3 ft. 6 in. behind the board?

5. A man climbs 16 feet up a 20-foot ladder, placed against a wall with the top 16 feet from the ground. How far are his feet from the ground?

6. In order to find the height of a tower a man sets up a pole 10 feet high 190 feet from the foot of the tower. He then walks away from the tower until he sees the top of the tower in line with the top of the pole; he is now 10 feet from the pole. His eye is 5 ft. 6 in. from the ground. How high is the tower?



7. A, B, C are three landmarks on a map in a straight line. A is on the 100-ft. contour, B on the 200-ft. and C on the 500-ft. AB is 1 mile and BC 3 miles measured from the map. Determine whether A is visible from C.

8. A, B, C are three landmarks in a straight line down a slope, A on the 300-ft. contour, B on the 200-ft., and C on the 150-ft. Measured on the map AB is 500 yd. and BC 250 yd. Determine whether the slope ABC is uniform.

9. A straight sight-seeing railway of uniform gradient has to pass by three points A, B, C respectively 50 ft., 150 ft., 350 ft. above sea-level. The distances between the points on a map are 140 yd. and 300 yd. respectively. Is it necessary to excavate or make an embankment at C and to what depth?

The figures shown are portions of a bridge framework. The measurements are in feet.

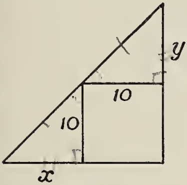


Fig. 8.4

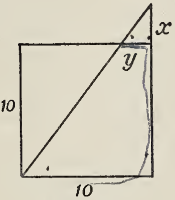


Fig. 8.5

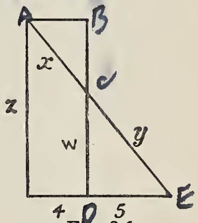


Fig. 8.6

✓

10. Fig. 4. Square and right triangle.

(a) Letter the figure. How many triangles are similar in this figure? What are the corresponding angles and sides?

(b) (i) If $x=5$, find y;

(ii) Express y in terms of (as a function of) x.

11. Fig. 5. Square and right triangle. As in ex. 10.

12. Fig. 6. Rectangle and right triangle.
(a) As in ex. 10(a);
(b) (i) If $z=10$, find w,
 (ii) Express w as a function of z;
(c) (i) If $x=3$, find y,
 (ii) Express y as a function of x;
(d) (i) $w=12$, find y,
 (ii) Express y as a function of w;
(e) Let $t=x+y$.
 (i) If $t=15$, find x,
 (ii) Express x as a function of t.

Three Dimensions

A right pyramid stands on a base 10 *ft. square and is* 15 *ft. in height. It is cut by a plane* 10 *ft. above the base and parallel to the base. Calculate the sides of the figure formed.**

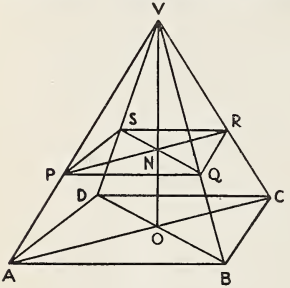


Fig. 8.7

PQRS is the plane cutting the pyramid. **OV** = 15 ft., **ON** = 10 ft.

$\therefore$ **NV** = 5 ft.

PQRS is a figure similar to **ABCD**, that is, a square.

Let x ft. be its side.

$\triangle$s **VNP** and **VOA** are similar,

$$\therefore \frac{AO}{PN} = \frac{OV}{NV} = \frac{15}{5}.$$

$\triangle$s **PNQ** and **AOB** are similar, $\therefore \dfrac{AO}{PN} = \dfrac{AB}{PQ} = \dfrac{10}{x}.$

Hence $\dfrac{15}{5} = \dfrac{10}{x}$, or $x = \dfrac{10}{3}$, $\therefore$ side of section $= 3\frac{1}{3}$ ft.

13. The base of a right pyramid is a square of side 12 cm. and a section parallel to the base and 8 cm. from it has a side of 5 cm. Find the height of the whole pyramid. (Use x and form an equation.)

---

*Any two squares are similar since each square may be divided into two equal right isosceles triangles.

14. The height of a cone is 20 in. and the base radius 8 in. Calculate the radius of a right section 7 in. from the vertex.

15. A point is taken dividing the slant height of a cone in the ratio of 2 : 3, measured from the vertex. Through this point a section is cut parallel to the base. If the radius of the base is 18 in., what is the area of the section?

B. SIMILAR TRIANGLES

16. **ABC**, **PQR** are two equiangular triangles.

(1) If $a=2$, $b=3$, $c=4$, $p=12$, what are q, r?

(2) If $p=4$, $q=5$, $r=6$, $a=3$, what are b, c?

(*Example.*) *If* $a=3$, $b=4$, $c=2$, $q=6$, *find* p.

Draw a figure and mark in the given lengths.

Since Δ**ABC** *is equiangular with* Δ**PQR**,

$$\therefore \frac{a}{p}=\frac{b}{q},$$

$$\frac{3}{p}=\frac{4}{6}.$$

Multiplying both sides by $6p$, $18=4p$,

$$p=\frac{18}{4}=4.5.$$

17. **ABC** is a triangle with sides $a=3$ in., $b=5$ in., $c=6$ in. Calculate the sides of an equiangular triangle which is to have 10 in. for its longest side.

18. Two right-angled triangles have each an angle of 53°. Their hypotenuses are 75 mm. and 45 mm. and the larger has the other two sides 45 mm. and 60 mm. Calculate the remaining sides of the smaller.

19. Are the following pairs of triangles similar?

(i) Sides 3, 6, 5 cm. and 2, 4, $3\frac{1}{3}$ cm.;
(ii) Sides 1.5, 2, 3 in. and 3, 4, 6 in.;
(iii) Sides 12, 20, 10 ft. and 3, 5, 2.5 in.;
(iv) Sides 2.5, 4.5, 3 cm. and 1.25, 2, 1.5 cm.

20. ABC is a rod, 12 cm. long, movable about the fixed point B. AB is 5 cm. and BC 7 cm. long. The rod is turned to a new position A′BC′. If AA′ = 4 cm., what is CC′? (The principle of proportional compasses.)

21. Repeat ex. 2 with QR = 5.8 cm., RP = 7.7 cm., PQ = 8.3 cm., VW = 3.8 cm.

22. A triangular face of a small crystal is viewed in a microscope and its sides appear to be 4.2 cm., 1.9 cm., 3.6 cm. If the longest side is actually 2 mm., what are the lengths of the other two sides?

Example. ABC *is a triangle,* DE *parallel to* AC *cuts* AB, BC *in* D, E*; if* AB = 20 *cm.,* AC = 12 *cm.,* BD = 8 *cm., calculate* DE.

In △*s* ABC, DBE, *since* DE *is* || *to* AC,

∠BAC = *cor.* ∠BDE,
∠ACB = *cor.* ∠DEB,
∠B *is common,*
∴ *the triangles are equiangular,*

$$\therefore \frac{BA}{BD} = \frac{AC}{DE}.$$

Let DE = x *cm.*

$$\therefore \frac{20}{8} = \frac{12}{x},$$

$$20x = 96, \qquad x = \frac{96}{20},$$

∴ DE = 4.8 *cm.*

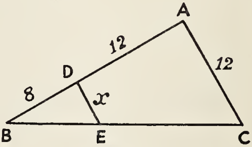


Fig. 8.8

23. In a triangle ABC, DE is drawn parallel to AC, as in fig. 8.

(i) If $BE = 4$ in., $EC = 3$ in., $ED = 1.5$ in., calculate AC;

(ii) If $BC = 16$ in., $CA = 9$ in., $ED = 4$ in., calculate BE.

24. A triangular steel wedge is 3 in. long and 1.5 in. wide at its base. It is driven 0.75 in. into a block of wood. What is the width of the hole made?

25. Two equal rods 14 in. long are bolted together 5 in. from one end of each and are placed with the ends of their longer portions 10 in. apart. What is the distance between their other ends?

26. The figure is a cross-section of a gable roof with an attic room.

(a) Letter the figure. How many sets of similar triangles are there? What are the corresponding sides and angles?

(b) Find the altitude of the equilateral triangle;

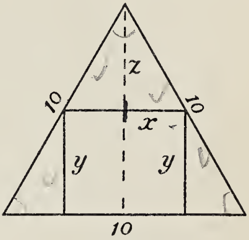


Fig. 8.9

(c) (i) $x = 3$. Find z and hence y;

(ii) Express z as a function of x;

(iii) Express y as a function of x;

(iv) Express the area of the rectangle as a function of x;

(v) Between what limits does x lie? y? z? the area of the rectangle?

Three Dimensions

27. A street light is 24 ft. above the sidewalk of a street, the street being 30 ft. wide. A man 6 ft. tall is walking along the opposite side of the street starting from the corner opposite the lamp.

(a) Draw a figure showing the position of the man and his shadow at the start;

(b) (i) Draw a figure showing not only the start, but a later position of the man and his shadow,

(ii) Letter the figure. How many pairs of similar triangles are there?

(iii) What is the locus of the tip of the shadow as the man walks along the street? Why?

(c) Find the length of the shadow (i) at the outset; (ii) when the man is 50 ft. from the corner.

6. Similar quadrilaterals.

(i) *Any two squares are similar.*

To prove this draw one pair of corresponding diagonals. Each square consists of two congruent right triangles. Theorem 29 shows that corresponding pairs of triangles in the two squares are similar. Hence the squares are similar.

(ii) *Any two rectangles with corresponding sides in proportion are similar.*

The method of proof is as in (i).

(iii) Are two parallelograms with corresponding sides in proportion similar? A simpler related question is: Are two parallelograms with corresponding sides equal, congruent? Think of the parallelograms as made with toy construction strips.

It is not enough then for quadrilaterals to be similar to ask that corresponding sides be proportional.

(iv) Would two quadrilaterals with corresponding sides and one diagonal proportional be similar? In this instance what about corresponding angles?

EXERCISES. Similar Quadrilaterals

1. Prove (i) and (ii).

2. Consider (iv) for (a) a pair of parallelograms; (b) a pair of quadrilaterals.

3. (a) Can you draw a pair of equiangular parallelograms which are not similar?

(b) Equiangular trapeziums?

(c) Are the trapeziums in fig. 10 similar? Are they equiangular?

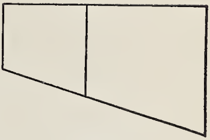


Fig. 8.10

4. (a) Are similar parallelograms equiangular?

(b) Trapeziums?

(c) Quadrilaterals?

5. The section of a railway cutting is an isosceles trapezium 100 ft. wide at the top and 30 ft. at the bottom, and its vertical depth is 20 ft. What is the width of the cut 6 ft. from the bottom?

6. Section of a dam, distance in ft. Fig. 11.

(i) Letter the figure;

(ii) Find y when $x=25$; $x=30$;

(iii) Express y as a function of x;

(iv) For what values of x may this formula be used?

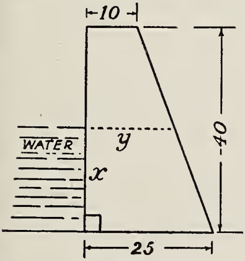


Fig. 8.11

7. Necessary and sufficient conditions.

You saw in §5 that if a triangle T′ is to be similar to a given triangle T certain conditions or restrictions are necessarily placed on T′. These were:

(i) If T′ is similar to T then a necessary condition on T′ is that it be equiangular with T.

(ii) If T′ is similar to T then a (second) necessary condition on T′ is that the sides of T′ be proportional to those of T.

You also saw that either of the restrictions (i) or (ii) guaranteed the similarity of T and T′. Stated formally these conditions are:

(iii) A sufficient condition for T′ to be similar to T is that T′ be equiangular with T;

(iv) A sufficient condition for T′ to be similar to T is that the sides of T′ be proportional to those of T.

Further, we may collapse (i), (iii) and (ii), (iv) together as follows:

(v) A necessary and sufficient condition that T′ be similar to T is that T′ be equiangular with T;

(vi) A necessary and sufficient condition that T′ be similar to T is that the sides of T′ be proportional to those of T.

Conditions then may be (a) necessary; or (b) sufficient; or (c) necessary and sufficient. They may also be (d) necessary but not sufficient; (e) sufficient but not necessary.

EXAMPLES

1. The drinking of water in some form or another is a necessary condition for your physical existence, but it is not a sufficient condition.

2. In North America an income of $3,000 a year for a family of two parents and one child is sufficient for their sustenance. It is not however a necessary basis.

3. A large suspension bridge built recently was designed so that the span under wind pressure could swing out of line up to 13.5 in. at the centre of the bridge. Shortly after its completion an unusually heavy gale swept the channel and the bridge swayed 8 in. at centre. The action of the storm showed that (i) an allowance of more than 8 in. was necessary if the bridge was to withstand storms; (ii) the allowance of 13.5 in. was sufficient to meet the strain of any storm likely to occur.

EXERCISES. Necessary and Sufficient Conditions

1. Complete the sentences by filling in phrases such as: "a necessary condition for" (an accompaniment of), "necessary to," "a sufficient condition for" (a guarantee of), "sufficient for".

(a) "Three strikes and you're out."

(i) being out is........................three strikes,
(ii) Three strikes is........................being out;

(b) Laugh and grow fat. According to this statement, Laughter is........................growing fat;

(c) When the temperature of a gas is constant its volume decreases as the pressure increases.

(i) A decrease in volume is........................an increase in pressure, under constant temperature;
(ii) An increase in pressure is........................a decrease in volume, under constant temperature;

(d) A boastful boy is probably suffering from feelings of inferiority. According to this statement feelings of inferiority are probably........................boastfulness;

(e) (1) If **A** then **B**. **B** is........................**A**. (2) **Q** only if **P**. **P** is........................**Q**.

2. (i) In ex. 1(a) is three strikes necessary to being out?

(ii) According to the statement in 1(d) is inferiority in a person a probable guarantee of boastfulness in that person?

(iii) Give examples of 1(e).

3. A certain city council stated that a tax rate of 52 mills on the dollar was necessary in view of the city's debt and budgeted expenditure. They also said that they were satisfied this rate would be sufficient to meet these obligations and expenditures. In view of these statements, what is your opinion of (i) a rate of 50 mills; (ii) a rate of 55 mills? Use the words "necessary" and (or) "sufficient".

4. A physician states that if a patient shows symptoms **A**, **B**, **C**, **D** then he is "satisfied" that the patient has disease **Q**, but that even if **A**, **B** and **D** show but not **C**, then he is satisfied

that the disease is not Q. Use the words "necessary" and "sufficient" to discuss this statement.*

5. State fully a necessary and sufficient condition that a triangle be a right triangle. Do you know any other necessary and sufficient condition for a right triangle? State it fully.

6. Is the following condition a necessary condition for similar quadrilaterals; a sufficient condition; both necessary and sufficient? The quadrilaterals:

(a) are equiangular;
(b) have corresponding sides proportional;
(c) have corresponding sides and one diagonal proportional;
(d) have corresponding sides proportional and one angle equal.

VIII. Similar Figures

A. PROPORTIONAL DIVISION

THEOREM 28. **If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.**

COROLLARY. **If a straight line is drawn parallel to one side of a triangle, the two triangles so formed have their corresponding sides in proportion and their corresponding angles equal.**

THEOREM 29. (CONVERSE OF THEOREM 28.) **If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.**

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*"Satisfied" means "reasonably certain". There can be no absolute certainty in the application of reasoning to natural phenomena or to human affairs. When geometry is applied to the making of a bridge truss, for example, we can never say that two members of the truss are exactly perpendicular. All we can say is that the adjacent angles are equal to the degree of precision required for such structures.

B. Similar Triangles

Terms: similar figures.

Theorem 30. **If two triangles are similar, corresponding sides are proportional.**

Theorem 31. (Converse of Theorem 30.) **If two triangles have corresponding sides proportional they are similar.**

Corollary. (Right triangles.) **If two right triangles have two pairs of corresponding sides proportional the ratio of the third pair of sides is equal to that of the other pair, and the triangles are similar.**

Theorem 32. **If two triangles are similar, corresponding vertex angles are equal, i.e., the triangles are equiangular.**

Theorem 33. (Converse of Theorem 32.) **If two triangles are equiangular the triangles are similar.**

Corollary. (Right triangles.) **Two right triangles with one pair of corresponding acute angles equal are similar.**

Theorem 34. **If two triangles have an angle of the one equal to an angle of the other and the sides about these equal angles proportional, the triangles are similar.**

C. Similar Quadrilaterals

The proofs of the theorems are left to the student; hints are not given in chapter X. The theorems have already been discussed in the exercises of §6 and §7.

Theorem 35. **If two quadrilaterals are similar, corresponding sides are proportional.**

Theorem 36. **If two quadrilaterals are similar, corresponding angles are equal.**

Theorem 37. **If two quadrilaterals have corresponding sides and one pair of corresponding diagonals in proportion, the quadrilaterals are similar.**

Theorem 38. **If two quadrilaterals have corresponding sides proportional and have one pair of corresponding angles equal, the quadrilaterals are similar.**

CHAPTER IX

SIMPLE TRIGONOMETRY

1. The trigonometric ratios.

Consider fig. 1.

A set of axes are drawn with the same unit of measurement for both axes. A half-line t is drawn through O as shown, and any point P distinct from O is taken on t. MP is drawn perpendicular to the x-axis, so that P has co-ordinates $x = \text{OM}$, $y = \text{MP}$. The distance of P from O is denoted by $r = \text{OP}$.

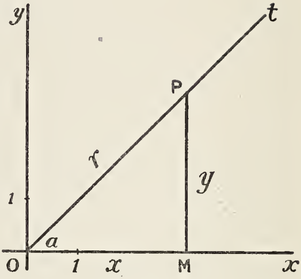


Fig. 9.1

What happens as P moves along t? The next figure shows two positions P_1 and P_2 with co-ordinates (x_1, y_1) and (x_2, y_2), their distances from O being r_1 and r_2.

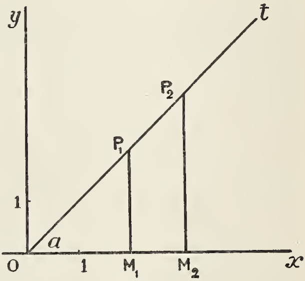


Fig. 9.2

The pair of right triangles OM_1P_1 and OM_2P_2 are evidently similar, and hence from fig. 3 we find,

$$\frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2} \text{ or } \frac{y_1}{r_1} = \frac{y_2}{r_2}$$

$$\frac{OM_1}{OP_1} = \frac{OM_2}{OP_2} \text{ or } \frac{x_1}{r_1} = \frac{x_2}{r_2}$$

etc.

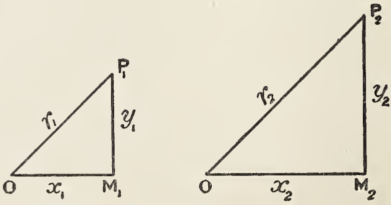


Fig. 9.3

These equations in effect mean that as P moves along t in fig. 1 the ratios

$$\frac{\text{MP}}{\text{OP}} = \frac{y}{r}, \quad \frac{\text{OM}}{\text{OP}} = \frac{x}{r}, \text{ etc.,}$$

remain the same for all positions of P. In other words, the ratio of any two of the three variable numbers x, y, r is the same for all positions of P on t.

As long as t stays in one position any one of the ratios remains unchanged. As soon as t moves about O, however, that ratio is altered. Hence each of the ratios formed from x, y, and r is solely dependent on $\angle \text{MOP} = \angle a$, i.e., each ratio is a function of $\angle a$.

EXERCISES. The Trigonometric Ratios

1. Reproduce fig. 1 for $a = 30°$. Find y and r by measurement or by computation when $x = 2, 3, 4, 5$. Compute the ratios $\frac{x}{r}$, $\frac{y}{r}$, $\frac{y}{x}$ for each of these values of x.

2. As in ex. 1 for (i) $a = 45°$, (ii) $a = 60°$.

3. (a) Are x, y, r related? How?

(b) As a varies, which is the largest of x, y and r? Always?

(c) How many possible ratios are there, formed from x, y and r?

(d) How small do you think $\frac{y}{r}$ can be? How large $\frac{x}{r}$?

2. The trigonometric functions of an angle.

The ratios of §1 are very old, having been used in principle by *Hipparchus* (c. 140 B.C.) in making a catalogue of stars.

The principal ratios are $\frac{y}{r}$, $\frac{x}{r}$, and $\frac{y}{x}$ and these functions of the angle *a* are given the following names:

$$\frac{y}{r} = \text{the sine of } \angle a,$$

$$\frac{x}{r} = \text{the cosine of } \angle a,$$

$$\frac{y}{x} = \text{the tangent of } \angle a.$$

or, more briefly,

$$\sin a = \frac{y}{r},\ \cos a = \frac{x}{r},\ \tan a = \frac{y}{x}$$

The other three remaining ratios, the reciprocals of these, also have names:

$$\frac{r}{y} = \text{the cosecant of } \angle a,$$

$$\frac{r}{x} = \text{the secant of } \angle a.$$

$$\frac{x}{y} = \text{the cotangent of } \angle a.$$

or, more briefly,

$$\operatorname{cosec} a = \frac{r}{y},\ \sec a = \frac{r}{x},\ \cot a = \frac{x}{y}.^{*}$$

Since these functions are extensively used for computing, tables have been prepared giving their values for various angles.† These tables are easy to use and practice with them is given in the exercises which follow.

---

*You need not learn these; they will not be used in this chapter.

†See the tables at the end of the book.

A three-figure table may be read to the tenth of a degree (6 minutes), whereas a four-figure table may be read to the nearest minute. Three-figure data only will be used in exercises.

EXERCISES. The Trigonometric Functions

1. (i) Draw the triangle shown in fig. 4 to scale and measure *a* with a protractor.

(ii) Find *a* to the nearest degree from a table of sines ($\sin a = 4/5 = 0.8$). Check (i).

(iii) Find *a* from a table of cosines; a table of tangents.

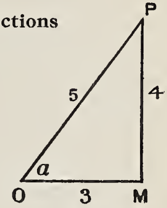


Fig. 9.4

2. (i) Compute the missing side to three figures in each of the following triangles.

(ii) As in ex. 1 for each triangle.

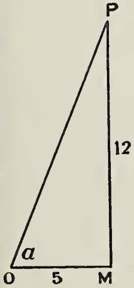


Fig. 9.5

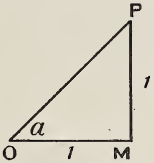


Fig. 9.6

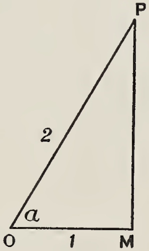


Fig. 9.7

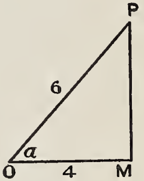


Fig. 9.8

3. (a) Make a scale drawing on the pattern of fig. 1 and find by measurement $\sin a$, $\cos a$, $\tan a$ for:

(i) 35°, (ii) 50°, (iii) 15°, (iv) 80°.

Use large triangles for the measurement of x, y, and r.

(b) Check by tables the values obtained for the three functions.

4. Use tables to verify the following, *a* being given:

a:	18°54′	35°	46°12′	80°0′	15°12′	53°18′	75°48′
sin a:	0.324	0.574	0.722	0.985	0.262	0.802	0.969
cos a:	0.946	0.819	0.692	0.174	0.965	0.598	0.245
tan a:	0.342	0.724	1.043	5.671	0.272	1.342	3.952.

5. As in ex. 4, sin *a* being given:

sin a:	0.469	0.127	0.837	0.927
a:	28°0′	7°18′	56°48′	68°0′ .

6. As in ex. 4, cos *a* being given:

cos a:	0.153	0.551	0.056	0.968
a:	81°12′	56°36′	86°48′	14°30′.

7. As in ex. 4, tan *a* being given:

tan a:	0.682	2.205	35.8	0.100
a:	34°18′	65°36′	88°24′	5°42′.

8. Use tables to draw the graphs of (i) sin *a*, (ii) cos *a*, (iii) tan *a* for angles *a* from 0° to 90°. *a* should be shown on the horizontal scale with the function on the vertical scale. Enough points should be plotted to obtain smooth curves. Label each graph clearly with its name.

9. You will have noticed the rather odd names of the functions. The sine function got its name in India (c.510), being the word *jīva = chord.* For this Arabic scholars (c.800) used their word *gib = fold.* Later, when the scholars of the middle ages in Europe obtained access to Arabian writings (c.1200), the latin word *sinus = fold* was used, whence the word *sine.*

Make a list of five words of any kind which seem curious words and try to trace their origin.

3. Special angles.

There are three important special acute angles, namely 30°, 45°, 60°. The values of the functions for these angles may be found from fig. 1 as follows:

(*a*)—30°.

Δ **OMP** is half of an equilateral triangle. Draw Δ **OMP** to scale with **OP** $=2$ in. For any such triangle $r=2y$ and $x=\sqrt{3}y$, so that

$$\sin 30° = \tfrac{1}{2}, \ \cos 30° = \sqrt{3}/2, \ \tan 30° = 1/\sqrt{3};$$

(*b*)—45°.

Δ **OMP** is an isosceles right triangle, with $x=y$ and $r=\sqrt{2}x$.

$$\sin 45° = \frac{1}{\sqrt{2}}, \qquad \cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1;$$

(*c*)—60°.

Δ **OMP** is again half of an equilateral triangle with $r=2x$ and $y=\sqrt{3}x$.

$$\sin 60° = \sqrt{3}/2, \quad \cos 60° = \tfrac{1}{2}, \quad \tan 60° = \sqrt{3}.$$

These special values should be learned from the triangles until they are familiar. It should not be forgotten, however, that such functions may be found directly from tables.

There are two other special angles which occur frequently in practice.

(*d*)—0°.

Here **P** is on the x-axis, so that $y=0$, with $r=x$.

$$\sin 0° = 0, \quad \cos 0° = 1, \quad \tan 0° = 0;$$

(*e*)—90°.

P is on the y-axis and $x=0$ with $r=y$.

$$\sin 90° = 1, \ \cos 90° = 0, \ \tan 90° \text{ is meaningless.*}$$

---

*As t (fig. 1) swings toward the vertical, $\angle a$ approaches 90°, and the tangent function becomes larger and larger without limit. When $\angle a = 90°$, the tangent function is without meaning.

EXERCISES. Special Angles

1. Complete the table, using $\sqrt{2} = 1.414$, $\sqrt{3} = 1.7321$.

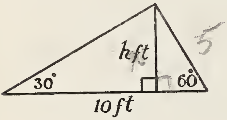
Show results to four decimal places and check your answers with trigonometric tables.

a:	30°	45°	60°
$\sin a$:	0.5		
$\cos a$:			0.5
$\tan a$:		1	

2. State, by inspection, what a is in each instance:

(a) $\sin a = 0$; (g) $\sin a = \frac{1}{2}$;
(b) $\cos a = 0$; (h) $\cos a = \frac{1}{2}$;
(c) $\tan a = 0$; (i) $\tan a = \sqrt{3}$;
(d) $\sin a = 1$; (j) $\sin a = \sqrt{3}/2$;
(e) $\cos a = 1$; (k) $\cos a = 1/\sqrt{2}$;
(f) $\tan a = 1$; (l) $\tan a = 1/\sqrt{3}$.

3. Find the height of an isosceles triangle base 10 cm. with base angle (a) 60°, (b) 45°, (c) 30°. Measure the height in each case.



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4. The figure shows a wooden framework, of width 10 ft. Find its height h ft.

5. A small bridge truss in the shape of an isosceles triangle, base angle 30°, is 6 feet high. How long is it? Measure its length.

6. Find the area of the parallelogram with sides 7.53 cm., 4.96 cm. and angle 60°. Measure the area.

7. The chord AB of a circle with radius r subtends an angle of 120° at the centre of the circle. Find the length of the chord and its distance from the centre of the circle.

4. The angles in a right triangle.

Given any right triangle. What are the trigonometric functions of its angles?

Let Δ UVW be given with W = 90°.* Introduce axes as in figs. 9 and 10, (i) with origin at U, (ii) with origin at V.

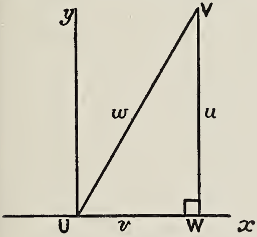


Fig. 9.9

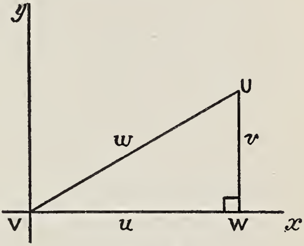


Fig. 9.10

The definitions of the functions (§2) may be applied at once to give

$$\sin \mathsf{U} = \cos \mathsf{V} = \frac{u}{w}, \quad \cos \mathsf{U} = \sin \mathsf{V} = \frac{v}{w}, \quad \tan \mathsf{U} = \cot \mathsf{V} = \frac{u}{v} \cdot$$

Note also that since W = 90°,

$$\sin \mathsf{W} = 1, \quad \cos \mathsf{W} = 0, \quad \tan \mathsf{W} \text{ is meaningless.}$$

The following meanings for the functions of U (and V) are helpful, provided it is remembered that they are true only for acute angles in a right triangle.†

$$\sin \mathsf{U} = \frac{\text{side opposite to U}}{\text{hypotenuse}} = \frac{u}{w},$$

$$\cos \mathsf{U} = \frac{\text{side adjacent to U}}{\text{hypotenuse}} = \frac{v}{w},$$

$$\tan \mathsf{U} = \frac{\text{side opposite to U}}{\text{side adjacent to U}} = \frac{u}{v} \cdot$$

---

*In this and the next paragraph capital letters U, V, W, etc., are used for the angles in a right triangle, with small letters u, v, w, etc., for the sides opposite to these angles.

†See §7.

You will have noted above that cos V = sin U, where U = 90° − V is the complement of V. We have thus verified for an acute angle that

The cosine of any angle is equal to the sine of its complementary angle,

and hence that

The sine of any angle is equal to the cosine of its complementary angle.

Stated in symbols, these theorems become:

$$\cos a = \sin (90° - a),$$
$$\sin a = \cos (90° - a).$$

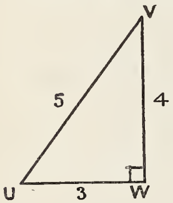
It is also true that

$$\cot a = \tan (90° - a), \qquad \operatorname{cosec} a = \sec (90° - a),$$
$$\tan a = \cot (90° - a), \qquad \sec a = \operatorname{cosec} (90° - a).$$

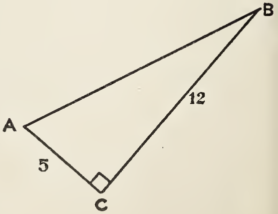
The word "cosine" means, of course, complementary-sine-i.e., $\cos a = \sin$ (Compl. of a). Likewise for the words "co, tangent" and "cosecant".

EXERCISES. Acute Angles

Check the statements in each of the following:

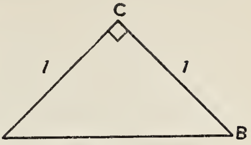


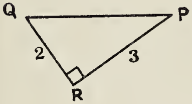
1. sin U = 4/5, sin V = 3/5, cos U = 3/5, tan V = $\frac{3}{4}$;



2. tan A = 12/5, cos B = 12/13, cos A = 5/13, sin B = 5/13;

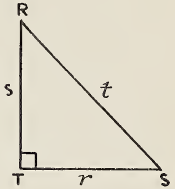
3. $\sin A = \frac{1}{\sqrt{2}}$, $\sin B = \frac{1}{\sqrt{2}}$,
$\tan B = 1$, $\cos A = \cos B$;

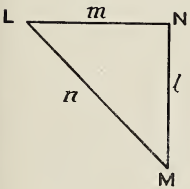




4. $\cos P = \frac{3}{\sqrt{13}}$, $\tan Q = \frac{3}{2}$,
$\tan P = 1/\tan Q$;

5. $\sin R = \frac{r}{t}$, $\tan S = \frac{s}{r}$, $\sin S = \cos R$;





6. $\tan M = m/l$, $\sin L = l/n$,
$\cos M = \sin L$.

5. The right triangle.

You have seen in chap. II how right triangle problems were solved by the use of scale drawings. This method is rapid and often quite sufficient. The accuracy of the method is rather limited, however, and numerical methods are frequently needed. The theory of such methods is based on the trigonometric functions.

Example 1

*A calibrated staff, graduated in inches, is set vertically on the ground and the length of its shadow is measured. The height of the staff is 3 ft. 4 in. and the shadow is 3 ft. 9 in. long. What was the altitude of the sun?**

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*This was done with a broomstick, notched with a pen-knife for length, with a nail driven in one end. A string and a weight made a plumb-line to ensure that the staff was vertical.

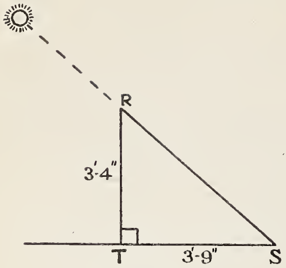


Fig. 9.11

It is required to find S in fig. 11. Since $r=3.75$ ft. and $s=3.33$ ft., then $\tan S=3.33/3.75=0.888$.

By tables $S=41°36'$ or $42°$ to the nearest degree.

The altitude of the sun was 42°.

Check: Draw the data to scale.

Example 2

In a land survey two stations A *and* B *are each placed* 205 *yd. from station* O. *The bearings of* A *and* B *from* O *are* N. 40°30′ E. *and* N. 74°36′ W. *Find* AB.

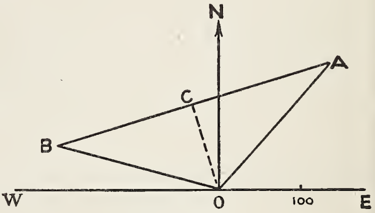


Fig. 9.12

The data are drawn to scale as in fig. 12. With the unit shown AB can be estimated as about 350 yd.

Let OC be drawn perpendicular to AB.

$\therefore BC=OB\sin BOC.$

$AB=2\,(205)\sin 57°33'.$

$=410\,(0.844)=346$ yd.

The distance between the stations A and B is 346 yd.

EXERCISES. The Right Triangle

1. ΔABC is given with $C=90°$. Find the required parts.

(a) $a=7$, $A=42°$; b;

(b) $c=42$, $B=62°$; a;

(c) $a=175$, $b=329$; A;

(d) $c=10.3$, $A=43°6'$; a, b. (Use $a=c\sin A$, $b=c\cos A$);

(e) $a=831$, $b=428$; B, c. (Use $c=a/\cos B$).

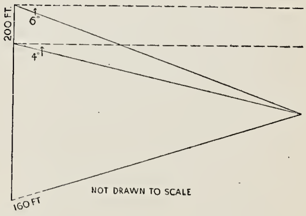
2–7. Ex. 5-10, §2, chap. II.

The following exercises were invented by a class of students as examples of the use of right triangles.

8. To find the height of a cliff or a steep hill. The angle of elevation at the beginning of the ground rise is 42°, and the angle of elevation 100 yd. back from the first point is 27°. Find the height of the hill.

9. A pyramid is 100 ft. wide at the base. The angle of inclination to the top is 15°06′ at a distance of 500 ft. from the centre of the pyramid. Find the height and slant height of the pyramid.

10. A man is standing on the roof of the Marine Building in Vancouver. He sights the opposite shore of Burrard Inlet, 6° below the horizontal. He goes down 200 ft. to the tenth floor of the building and sights the same point which is now 4° below horizontal. The building is 160 ft. from the shore. How wide is the inlet? How high is the building?



11. A vertical hole is drilled from a point A on a hill to B, the top of a 7 ft. coal seam. The hole is 160 ft. deep. A tunnel is started at C on the side of the hill, along the seam with its floor on a level with the bottom of the seam. A line from A to C makes an angle of 56° with the horizontal. How far will the tunnel have to be driven to reach the drill line AB when the seam is (a) horizontal, (b) inclined up from C at 10° to the horizontal? (Distance along the floor of the tunnel.)

6. Slope and inclination.

When a road is inclined at an angle of a to the horizontal we speak of a as the *inclination* of the road. The *slope* or *gradient* of the road will be tan a, being the ratio of "rise" to (horizontal) "run". For low grades, as on railways, $\tan a = \sin a$ approximately, so that the slope of the road can be measured as the vertical rise per unit distance along the road.

The slope of a road is often given as a percentage, viz., a 5% slope means that $\tan a = 0.05$.

Example

A road slopes upward, rising 100 *feet to the mile. At what angle is it rising?*

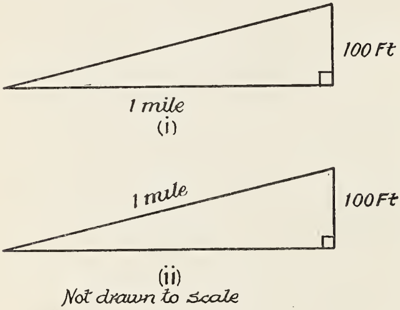


Fig. 9.13

$$\tan a = 100/5280$$
$$= 0.019,$$

so that $a = 1°6'$ to the nearest $6'$. In this instance $\sin 1°6'$ is also 0.019, so that either fig. (i) or (ii) may be used.

The road slopes upward with a grade of 1.9%.

EXERCISES. Slope and Inclination

✓1. A road rises uniformly 250 ft. for every half-mile of horizontal distance. Find its slope and inclination.

✓2. Mine spur, C.N.R. The spur is from Leyland (elev. 4,932 ft.) to Luscar (elev. 5,373 ft.) a track distance of 5.1 mi.

At what angle is the track rising? Find the gradient and the percentage gradient.*

3. Generally speaking, grades on main Alberta highways do not exceed 8%. What is the angle of inclination here if the grade is figured (a) as a slope, i.e., rise to horizontal run; (b) as rise to distance along the road.

4. Make a chart showing lines radiating from a point with percentage slopes of 0, 10, 20, . . . 100. Should the sine or tangent function be used?

7. The functions for any angle.

The definitions in §2 are so phrased that they may be used as they stand for any angle. Suppose as in fig. 14 we are given an angle of 110°.

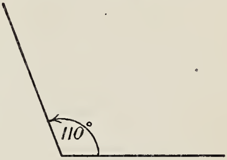


Fig. 9.14

What is meant by sin 110°? Fig. 1 drawn for this angle is shown in fig. 15.

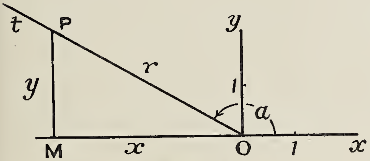


Fig. 9.15

The x-axis is put along one arm of the angle with the vertex of the angle as origin. **P** is any point distinct from **O** on t, the other arm of the angle. The reasoning of §1 may be followed through to show that the ratios x/r, y/r, y/x, etc., depend only upon $\angle a$. These ratios for any angle a are defined as in §2.†

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*The maximum gradient of this spur is 3%. The steepest grade on the C.N.R. occurs on the St. Martins' subdivision in New Brunswick between Hampton and St. Martins, the grade being 3.48% in an easterly direction and 3.78% in a westerly direction. You may wish to draw these grades to scale.

†We are not entitled to speak of **MP** as the side opposite $\angle a$, since this angle no longer belongs to Δ**OMP**.

You should note in fig. 15 that x is negative, with y positive and r also positive (r is taken to be positive for any angle). Hence sin 110° is positive while cos 110°, tan 110° are both negative.

The applications of these definitions for any angle may be mentioned briefly.

(i) These definitions lead to numerical methods for solving triangle problems of any sort (see chap. 3);

(ii) The methods of (i) may be generalized to give methods for solving triangle problems on the sphere, as required for surveying over large regions on the earth's surface;

(iii) (i) and (ii) are trigonometry in the literal sense—the theory and practice of triangle measurement. There are other fundamental applications of the trigonometric functions to any type of periodic behaviour, viz., orbital motion, wave motions of any sort (sound, heat, light, electricity).

(iv) The trigonometric functions play an important part in mathematical theory itself.

EXERCISES. Any Angle

1. Compute the functions from scale drawings to verify the results shown.

(a) $\sin 110° = 0.94$, $\cos 110° = -0.34$, $\tan 110° = -2.7$.
(b) $\sin 295° = -0.91$, $\cos 140° = -0.77$, $\tan 210° = 0.58$.

2. Show from a figure that:

(a) $\sin 120° = \sin 60°$, $\tan 150° = -\tan 30°$,
$\cos 210° = -\cos 30°$.
(b) $\tan 210° = \tan 30°$, $\cos 300° = \cos 60°$,
$\sin 315° = -\sin 45°$.

CHAPTER X

THE PROOF OF THEOREMS

"An assumption, Sir, not proof." (From the motion picture, "Stanley and Livingstone" (Fox Studios).)

PART I

1. Definitions.

The principal words in any careful statement of reasoning are of two kinds: (i) those left without specific meaning, e.g., "point" or "line" in geometry; (ii) those which are explicitly defined, e.g., a "circle" as a locus of points. (ii) are defined in terms of (i).

The meaning of words in reasoning is a matter of agreement or contract between persons, viz., we agree to say that line a is $\perp$ to line b if the adjacent angles made by a with b are equal. The agreement is binding upon the contracting parties unless a new agreement is made.

Example

For a long time people understood an angle to be an angle in a triangle, acute, right or obtuse. Later on the word "angle" had its meaning changed to include positive angles not exceeding 360° (see chap. III, §3). *Again, for vibration problems, i.e., wave-motion problems, an angle may be of any size, positive or negative.*

It is not always easy to define terms clearly. Joan says "Parallel lines are lines that do not intersect." Is this the agreed meaning for "parallel lines"? How should Joan's statement be amended?

Also, you have seen in chap. VIII how vague words in definitions make for difficulty.

EXERCISES. Definitions

1. How many definitions can you give for (i) triangle, (ii) parallelogram, (iii) circle? Do these agree?

2. Wireless signals from ships or aircraft are matters of agreement. What is S.O.S. in Morse? How is "A" defined in Morse? Give other examples of definitions by agreement.

3. (a) Does a dictionary define every word in it?

(b) Are there other words besides *point* and *straight line* not defined in the preceding chapters?

(c) What terms were undefined in your work in General Science?

4. Excerpt from a candy recipe: "When the syrup makes a soft ball in cold water remove from fire. . . . Let mixture stand until cool, beat, and pour into a buttered pan." What are the vague words in this instruction? How could you render them precise?

5. (a) A person is said to be normal. What could this mean?
(b) Can you define (i) electricity, (ii) self, (iii) nature?

6. (a) Give examples of words which mean the same thing;
(b) As in (a) for words which have more than one meaning.

2. Assumptions.

You have seen that some propositions (‡) were accepted without proof. These assumptions included Theorems 1, 2, 4(i), 8-10, 31 and Playfair's axiom.* The remaining propositions up to and including Theorem 34, together with the proofs of constructions 1-6 may be shown to depend primarily upon one or more of these assumptions.

---

*"Through a given point there is one and only one straight line which is parallel to a given straight line." An equivalent assumption appears in Euclid's *Elements* (c. 300 B.C.) as *Axiom 12*. "If a straight line meets two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles." *Euclid* (Home Univ. Lib.) p. 6.

Thus the proof of Theorem 38 depends upon Theorem 22 and upon knowing the area of a rectangle (an assumption not listed above). Theorem 22, in turn, requires Theorem 5 and Theorem 9. The proof of Theorem 5, however, requires Playfair's axiom. Hence Theorem 28 primarily depends upon (a) Playfair's axiom, (b) the area of a rectangle, (c) Theorem 9.

Again, Theorem 20 requires Theorem 18, Theorem 9, and Theorem 5 for its proof and hence depends upon (a) Playfair's axiom, (b) Theorem 9.

Other propositions (†) have been stated as assumptions, although these are capable of proof in terms of the primary assumptions (‡).

Must we accept the propositions (‡)? Euclid's parallel axiom (Playfair's axiom) was long a centre of controversy. Was it "true"? Could it be "proved"?

In chap. XIV you will see the beginnings of a geometry which rejects this assumption in favour of another. This change of assumptions (‡) leads of course to a geometry different from that of the preceding chapters.

EXERCISES. Assumptions

1. In each statement find (i) what is given (what are the data), (ii) the assumption or assumptions, (iii) the conclusion. Are different assumptions possible? How would these affect the conclusion?

(a) Bill knew that his dad had gone because the car was not in the garage.

(b) Mary knew the dress was perfect because Nell approved of it.

(c) The car stopped. George said, "Dad, I know it's the carburetor again."

(d) I know it is true because Jean told me.

(e) Mr. Aylesworth is not in the room because the class is so noisy.

(f) You could tell he was a deep thinker from the books he had in his room.

2. (a) Find assumptions which at one time were accepted by the majority but have since been rejected. Example: The heavier a body the faster it falls.

(b) If you can, state some of the effects of the rejections in (a).

3. Complete the following:

(i) Words may be defined or left..

(ii) Propositions may be proved or left..................................

(iii) An unproved proposition is called.................................

(iv) A proved proposition is called..

4. "That," said Jim, "is only an assumption." What is the effect of "only" on the rest of the statement? Is "only" ever merited in this connection? When?

5. "Most of the virtues attributed to 'civilization' did not exist before Christianity—they were then the dream of a few, they are now the conviction or assumption of the majority."*

(a) What is the definition of the word *conviction* in this statement?

(b) Apply this definition to some of your convictions.

(c) Do you agree with this usage of "conviction"?

6. "While the country owes no man a living, it does owe men the opportunity of earning a living, and it is my determination, if and when I am elected to lead the Government of Canada, to solve this problem as it must be solved if we are to preserve British democratic institutions as opposed to the freaky and irresponsible 'isms' that are proposed by others."

Find in the above quotation:

(a) Any words which need definition;

(b) Any assumptions, stated or implied;

(c) Try to define some of the words in (a). What persons might disagree with your definitions?

---

*Lord Stamp, *Christianity and Economics* (Macmillan, 1939), p. 69.

3. Theorems and their converses.

You will have seen that the statement of a proposition consists of given things (data or hypotheses) and required things (findings or conclusions).

EXAMPLES

1. THEOREM 8.

$\triangle RST$, $\triangle R'S'T'$

Data	*Findings*
1. $r = r'$,	1. $\angle R = \angle R'$,
2. $s = s'$,	2. $\angle S = \angle S'$,
3. $\angle T = \angle T'$.	3. $t = t'$.

2. THEOREM 11.

$\triangle RST$

Data	*Findings*
1. $r = s$.	1. $\angle R = \angle S$.

3. THEOREM 18.

Parallelogram XYZW

Data	*Findings*
1. $XY \parallel ZW$,	1. $XY = ZW$,
2. $XW \parallel YZ$.	2. $XW = YZ$,
	3. $\angle X = \angle Z$,
	4. $\angle Y = \angle W$,
	5. $\triangle XYZ \equiv \triangle YZW$,
	6. XZ bisects YW.

4. THEOREM 20.

$\triangle RST$

Data	*Findings*
1. $XR = XS$,	1. $YR = YT$.
2. $XY \parallel ST$.	

You are familiar with the idea of a converse theorem. Thus Theorem 12 was stated to be the converse of Theorem 11 since the data and findings of Theorem 11 interchange to give

Data	*Findings*
1. $\angle \mathsf{R} = \angle \mathsf{S}$.	1. $r = s$.

This interchange of one hypothesis with a single conclusion is the popular idea of a converse. A more general definition is possible, however.

DEFINITION. *A certain proposition is given with p hypotheses and q conclusions. A converse of the given proposition is obtained by interchanging any number of the hypotheses with an equal number of conclusions.*

(i) According to the definition, some propositions may have more than one converse.

(ii) Even if the given proposition is true, its converses need not be true; the proof of converses is necessary.

(iii) The meanings of hypothesis and conclusion in the above definition need to be made explicit. Thus in Theorem 18 as outlined above conclusion *5* implies conclusion *3*. If *5* is used in the interchange *3* must not be used simultaneously. Also conclusion *6* is a multiple conclusion and must be divided, viz., *6*. If O is the intersection of XZ and YW, OX = OZ and *7*. If O is the intersection of XZ and YW, OW = OY.

(iv) Try to re-state the above definition in the light of (iii).

What are the converse theorems obtainable from Theorem 8? Since one, two, or three hypotheses may be interchanged with one, two, or three conclusions there are in all 19 possible converse theorems. Why? (But see ex. 1.) Some of these converses are shown below. The interchange is indicated with an asterisk.

THEOREM 8 (C_1)

Data	*Findings*
1.* $\angle \mathsf{R} = \angle \mathsf{R}'$	1.* $r = r'$,
2. $s = s'$,	2. $\angle \mathsf{S} = \angle \mathsf{S}'$,
3. $\angle \mathsf{T} = \angle \mathsf{T}'$.	3. $t = t'$.

Theorem 8 (C_2)

Data	*Findings*
1. $r = r'$,	1. $\angle R = \angle R'$,
2. $s = s'$,	2. $\angle S = \angle S'$,
3.* $t = t'$.	3.* $\angle T = \angle T'$.

Theorem 8 (C_3)

Data	*Findings*
1.* $\angle R = \angle R'$,	1.* $r = r'$,
2.* $\angle S = \angle S'$,	2.* $s = s'$,
3. $\angle T = \angle T'$.	3. $t = t'$.

Theorem 8 (C_4)

Data	*Findings*
1. $r = r'$,	1. $\angle R = \angle R'$,
2. $s = s'$,	2.* $\angle T = \angle T'$,
3.* $\angle S = \angle S'$.	3. $t = t'$.

How many of the above theorems are false?

The example which follows is based on the oath administered to court witnesses in which the witness must swear to "tell the truth, the whole truth, and nothing but the truth."

Example

Arrange the following sentences as data and findings for possible theorems and their converses. State whether any of these propositions may be true or false.

(i) *John tells part of the truth,* (ii) *John tells the whole truth,* (iii) *John tells nothing but the truth.*

The propositions are shown in the table below. F = false, T = true, T or F = may be true or false. (1′) is the converse of (1), (2′) of (2), etc.

	Data	*Findings*	
(1)	(i)	(ii)	F
(1′)	(ii)	(i)	T
(2)	(i)	(iii)	T or F
(2′)	(iii)	(i)	T
(3)	(ii)	(iii)	T or F
(3′)	(iii)	(ii)	T or F
(4)	(i) (iii)	(ii)	F

Each proposition above should be stated in words, thus (2′):—"If John tells nothing but the truth, then John tells part of the truth."

Since (ii) guarantees (i), (i) is added to the oath only for emphasis. You should note that (ii) is no guarantee of (iii), nor is (iii) of (ii). The court rightly demands (ii) and (iii).

EXERCISES. Theorems and Converses

1. The data and findings of Theorem 8 refer to parts of triangles. These may be stated in words:

Data	*Findings*
1.2. Two sides equal,	1.2. Two angles equal,
3. Their included angle equal.	3. One side equal.

Use this method to show that only four distinct converse theorems for Theorem 8 may be found. How many of these are true? Are these numbered theorems in your list? Is each of the theorems a converse of the other?

2. Show that Theorem 20 has only one converse theorem.

3. Form converse theorems of Theorem 18 by interchanging

(a) two hypotheses and two conclusions;
(b) one hypothesis and one conclusion.

In (a) are all these converses true? Compare with Theorem 19. How many converses of Theorem 18 does Theorem 19 contain?

In (b) are any of the converses true?

4. (1) Show Theorem 32 with data and findings tabulated. (Use Theorem 30.)

(2) Is Theorem 33 a converse of Theorem 32?
(3) Can you find other converses? Are they true?

5. See Theorem 26 at the end of this chapter and the paragraph following: ". . . it is necessary to prove that I . . . and *conversely* that II . . ."

(1) Analyse I and II for data and findings.
(2) Was the word *conversely* properly used above?
(3) Is any other converse for I possible?

6. See Theorem 4 and Theorem 5. Are other converses for Theorem 4 possible?

7. Arrange any or all of following as *data* and *findings* for possible theorems and their converses. State whether any of these propositions may be true or false.

(a) (i) I am a musician, (ii) I play the piano well;
(b) (i) $x > o$, (ii) $y > x$, (iii) $y > o$;
(c) Either Chemistry or Physics is required for graduation. (i) Mary graduated, (ii) Mary passed Physics and Chemistry, (iii) Mary passed Physics or Chemistry.

8. *Raylex*, the soap of beautiful women. Arrange the following as in ex. 7. Which theorem would the advertiser prefer Jean to believe? Is it true?

(i) Some beautiful women use Raylex; (ii) Jean uses Raylex; (iii) Jean is beautiful.

9. *Buffet* cereal. As in ex. 8.

(i) Some famous athletes use Buffet; (ii) athletes are strong; (iii) John is strong; (iv) John is an athlete; (v) John eats Buffet; (vi) John is famous.

10. As in ex. 7 for the following:

(i) I am rich; (ii) I go on a motor trip; (iii) I like to swim; (iv) I catch cold; (v) I spend $400 every month; (vi) I go to Calgary; (vii) I stay at the Palliser Hotel.

4. Proofs.

Hints for proofs of the theorems used thus far are given in the second part of this chapter. Proofs should be written in the style indicated in earlier chapters. The proof of Theorem 28 is given below as a further example of this style. You will also find it useful to trace theorems back to the primary assumptions as already indicated in §2 for Theorem 28 and Theorem 20.

THEOREM 28. **If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.**

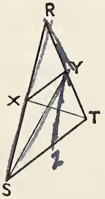


Fig. 10.1

Data ΔRST

(1) XY is parallel to ST.

To prove that $\frac{RX}{RS} = \frac{RY}{RT}$.

Construction. Join SY, TX.

Proof Δs RXY and RYS have the same altitude so that their areas are proportional to their bases.

$$\therefore \frac{\Delta RXY}{\Delta RYS} = \frac{RX}{RS} \qquad \text{(Theorem 22, corollary 3).}$$

$$\text{Also } \frac{\Delta RXY}{\Delta RXT} = \frac{RY}{RT}.$$

Δs **SXY** and **TYX**

$\left\{\begin{array}{l}\text{common base } XY \\ \Delta\text{s lie between the same } \|\text{s } XY \text{ and } ST.\end{array}\right.$

$$\therefore \Delta SXY = \Delta TYX. \qquad \text{(Theorem 22, corollary 2).}$$

$$\text{Also } \Delta SXY + \Delta RXY = \Delta TYX + \Delta RXY,$$

$$\therefore \Delta RYS = \Delta RXT,$$

$$\text{or } \frac{\Delta RXY}{\Delta RYS} = \frac{\Delta RXY}{\Delta RXT}.$$

Hence, as was to be proved,

$$\frac{RX}{RS} = \frac{RY}{RT}.$$

You should note that the data of Theorem 28 permit other shapes for Δ**RST** and other positions for **XY**. Draw a number of such figures.

Suppose that **RS** and **RT** are produced. Draw **XY** || to **ST** across these produced sides. Is it true that $\frac{RX}{RS} = \frac{RY}{RT}$? Why?

Can you produce the sides through **R**?

In proving any proposition you should draw figures illustrating *variability of the data*, as indicated above.

PART II

I. Angles, Parallels, Perpendiculars

A. ANGLES

No theorems.

B. TWO INTERSECTING STRAIGHT LINES

‡THEOREM 1. **The sum of any pair of adjacent angles formed by two intersecting straight lines is equal to two right angles.**

COROLLARY. **If any number of straight lines intersect in a point, the sum of all the consecutive adjacent angles is equal to four right angles.**

Hint: The corollary is true for two straight lines by Theorem 1. More than two lines increases the number of adjacent angles without altering their sum.

‡THEOREM 2. (CONVERSE OF THEOREM 1.) **If the sum of two adjacent angles is equal to two right angles, the exterior arms of the angles are in the same straight line.**

†THEOREM 3. **If two straight lines intersect, the vertically opposite angles are equal.**

Hint: Use small letters for angles. Write the equations obtained by Theorem 1. Subtract pairs of equations.

C. TWO STRAIGHT LINES CUT BY A THIRD STRAIGHT LINE

THEOREM 4. **Two straight lines are cut by a transversal.**
‡(i) **If a pair of corresponding angles are equal the two lines are parallel.**
†(ii) **If a pair of alternate angles are equal the two lines are parallel.**

†(iii) **If a pair of interior angles on the same side of the transversal are together equal to two right angles, the two lines are parallel.**

Hint for (ii): Use Theorem 3 and (i).

Hint for (iii): Use Theorem 1 and (i).

COROLLARY. **If each of two straight lines is perpendicular to a third straight line, the two lines are parallel to one another.**

Hint: See (iii).

ASSUMPTION. **(Playfair's Axiom.) Through a given point only one straight line can be drawn parallel to a given straight line.**

†THEOREM 5. (CONVERSE OF THEOREM 4.) **If two parallel straight lines be cut by a transversal,**

(i) **corresponding angles are equal;**

(ii) **alternate angles are equal;**

(iii) **the interior angles on the same side of the transversal are together equal to two right angles.**

Hint for (i): Apply the method of elimination. Either the angles are (a) equal, or they are (b) not equal. Assume (b). Then in fig. 2, $\angle x$ and $\angle y$ are unequal. Draw a line through L to make corresponding angles equal. Then apply Theorem 4 (i) and the above assumption to show that (b) is false and hence that (a) is true.

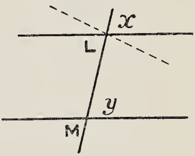


Fig. 10.2

(ii) and (iii) may be referred back to (i).

COROLLARY. **Straight lines which are parallel to the same straight line are parallel to each other.**

Hint: Consider corresponding angles (Theorem 5 (i) and Theorem 4 (i)).

D. RULER AND COMPASS CONSTRUCTIONS

The congruence theorems of **II** are required for these proofs.

Construction 1. To make an angle at **O** *equal to an angle at* **A.**

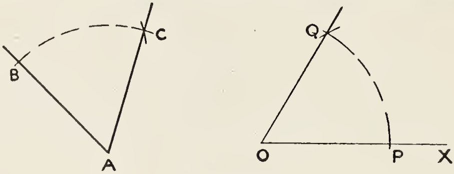


Fig. 10.3

Data ∠ **BAC** and point **O.**

Construction. Draw any line **OX** through **O.**

With centre **A** and any radius, draw a circle cutting the sides of the given angle at **B** and **C.**

With centre **O** and the same radius, draw a circle **PQ** cutting **OX** at **P.**

With centre **P** and radius **BC**, find **Q.**

Join **OQ.**

Then ∠ **QOP** = ∠ **BAC.**

Proof.

Hint: Join **BC** and **PQ** and prove Δ **ABC** ≡ Δ **OPQ.**

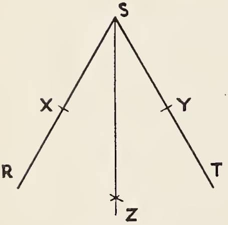


Fig. 10.4

Construction 2. To bisect an angle.

Data ∠ **RST.**

Construction. From **SR, ST** cut off equal lengths **SX, SY.**

With centres **X** and **Y** draw equal circles intersecting at **Z.**[*]

Join **SZ.** Then **SZ** bisects ∠ **RST.**

---

[*] It is always possible to draw such circles. These circles will intersect in another point **W.** A test of the accuracy of your construction is to see that **W** lies on **SZ.**

Proof.

Hint: Join **XZ**, **YZ**. Then prove $\Delta SXZ \equiv \Delta SYZ$.

Construction 3. To draw the perpendicular bisector of a straight line segment.

Data **RS**.

Construction. With centres at **R** and **S** draw equal circles intersecting at **X** and **Y**.

Join **XY** to cut **RS** at **Z**.

Then **XY** is the perpendicular bisector of **RS**.

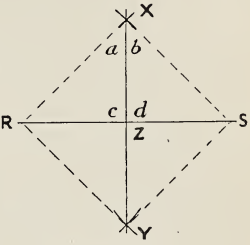


Fig. 10.5

Proof. Hint: You know that $RX = SX$ and $RY = SY$.

Hence show that $\angle a = \angle b$.

Since you know that $\angle a = \angle b$ and $RX = SX$ then $\angle c = \angle d$ and $RZ = ZS$.

Construction 4. To draw a perpendicular through a point to a straight line.

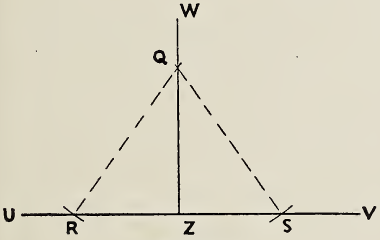


Fig. 10.6

Data A point **Q** and a straight line **UV**.

Construction. With centre **Q** draw a circle to intersect **UV** in two points **R** and **S**.*

Apply const. 3 to draw **ZW** the perpendicular bisector of **RS**, where **Z** is the mid-point of **RS**.

ZW is the required line.

Proof. Hints: **ZW** is perpendicular to the given line.

You need to prove that **Q** lies on **ZW**.

---

*Why is it always possible to draw such a circle?

There are two cases: (i) Q on UV, (ii) Q not on UV.
In (i) show that Q = Z by construction.
In (ii) join QZ and prove $\Delta RZQ \equiv \Delta SZQ$.

Construction 5. *To draw a straight line through a point parallel to a straight line.*

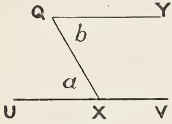


Fig. 10.7

Data A point Q and a straight line UV.

Construction. Join Q to any point X on UV. Apply construction *1* to draw QY, with Y on the opposite side of QX from U, so that $\angle b = \angle a$.
Then QY is parallel to UV.

Proof. No hint is necessary.

Construction 6. *From a given point to draw a straight line segment equal to a given straight line segment.*

See chap. V, §2, ex. 20, for the construction.

II. Side and Angle Properties of Triangles

A. Angle Properties

Theorem 6. **The sum of the angles of a triangle is equal to two right angles.**

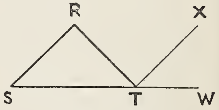


Fig. 10.8

Hints: Assume the theorem to be true. See Theorem 1. This suggests that one side be produced. Note now that Theorem 7 follows (assuming Theorem 6). Theorem 5 suggests that a parallel be drawn. The complete figure is shown (fig. 8). Now prove Theorem 6; start with the straight angle STW.

Corollary. **If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.**

Theorem 7. **If one side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.**

Hint: The proof of this theorem was discovered in the process of proving Theorem 6.

B. Congruent Triangles

‡Theorem 8. **If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles included by those sides equal, the triangles are congruent.** (S.A.S.)

Corollary. (Right triangles). **If two right triangles have the sides about the right angle of one triangle equal to the sides about the right angle of the other, each to each, the triangles are congruent.**

‡Theorem 9. **If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.** (A.S.A.) or (A.A.S.).

Corollary. (Right triangles.) **If two right triangles have an acute angle of the one equal to an acute angle of the other, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.**

‡Theorem 10. **If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.** (S.S.S.)

THEOREM 11. (Right triangles.) **If two right triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.**

(i) If the theorem of Pythagoras is assumed, the theorem follows from Theorem 10.

(ii) If the theorem of Pythagoras is not assumed, Theorem 12 may be called into play, by construction. The detail for (ii) follows.

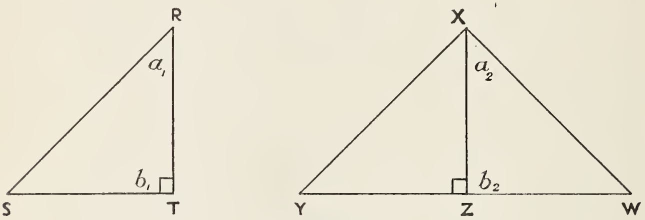


Fig. 10.9

Data Δs RST, XYZ.

(1) $\angle T = \angle Z = 1$ rt. ∠.
(2) hyp. RS = hyp. XY.
(3) RT = XZ.

To prove that ΔRST ≡ ΔXYZ.

Construction. At X and Z on the opposite side to Y make $\angle a_2 = \angle a_1$, $\angle b_2 = \angle b_1$ (= 1 rt. ∠)

Hints for proof: (a) Prove ΔXWZ ≡ ΔRST. (A.S.A.)
∴ XW = RS = XY.
(b) YZW is a straight line.
(c) ΔXYW is isosceles.
(d) ΔXYZ ≡ ΔXZW. (A.S.A.)

C. ISOSCELES TRIANGLES

THEOREM 12. **If two sides of a triangle are equal, the angles opposite to these sides are equal.**

Hint: If the theorem is assumed to be true, congruent triangles suggest themselves. Apply Theorem 8 by proper construction to prove the theorem.

Note: Theorem 12 may be used to prove Theorem 10 which we have hitherto assumed to be true. The method is somewhat like that indicated for Theorem 11 above, but the detail will not be given here.

THEOREM 13. (CONVERSE OF THEOREM 12.) **If two angles of a triangle are equal, the sides opposite to these angles are equal.**

Hint: Corollary, Theorem 9.

D. SIDE AND ANGLE INEQUALITIES

THEOREM 14. **If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.**

THEOREM 15. (CONVERSE OF THEOREM 14.) **If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.**

THEOREM 16. **Any two sides of a triangle are together greater than the third side.**

Hints: Suggestions for the proofs of these theorems have already been given in chap. IV, §3.

III. Side and Angle Properties of Quadrilaterals

THEOREM 17. **The sum of the angles of a quadrilateral is equal to four right angles.**

Hint: Draw a diagonal.

THEOREM 18. **If a quadrilateral is a parallelogram, then**

(i) **the opposite sides are equal;**

(ii) **the opposite angles are equal;**

(iii) **each diagonal divides the parallelogram into two congruent triangles;**

(iv) **the diagonals bisect each other.**

Hints: What is a parallelogram?

Prove (iii) first; (i), (ii), (iv) then follow.

Note: (i) may be proved directly if Theorem 18 (i) is assumed to be true for a rectangle. Can you prove (ii) directly?

THEOREM 19. (CONVERSE OF THEOREM 18.) **A quadrilateral is a parallelogram:**

(i) **when both pairs of opposite sides are equal;**
or (ii) **when both pairs of opposite angles are equal;**
or (iii) **when one pair of opposite sides are equal and parallel;**
or (iv) **when its diagonals bisect each other.**

Hints: What is a parallelogram?

For (i) and (iii) draw a diagonal and prove triangles congruent.

For (ii) use Theorem 17.

Prove triangles congruent in (iv).

IV. Triangles. Mid-point Theorems

THEOREM 20. **The straight line drawn through the mid-point of one side of a triangle parallel to another side bisects the third side.**

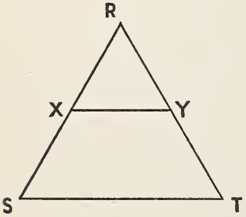


Fig. 10.10

Data $\triangle RST$.

(1) X is the mid-point of RS,
(2) XY is drawn parallel to ST to cut RT at Y.

To prove that $RY = YT$.

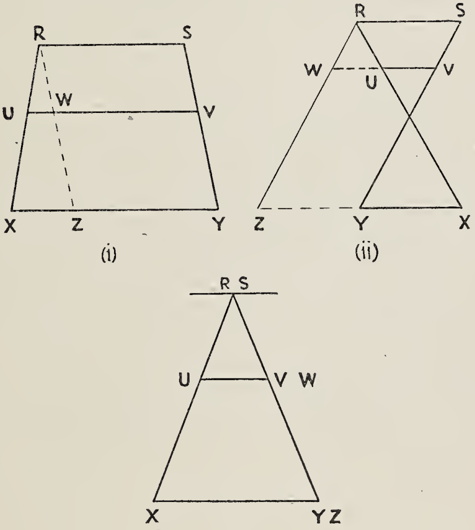
Construction.

Hint: A parallelogram may be constructed with one vertex either at X or T.

Corollary. **Given three parallel straight lines and two transversals of these lines. If one transversal is bisected by the parallels, so also is the other transversal bisected.**

Data Three parallel lines RS, UV, XY,
Two transversals RUX and SVY.
(1) RU = UX.

To prove that SV = VY.



(i) (ii)

(iii)

Fig. 10.11

Hints: The two main possibilities are shown in figs. (i) and (ii). Fig. (iii) is obtained when the transversals intersect on the uppermost parallel, and corresponds to Theorem 20.

In (i) and (ii) draw RWZ parallel to SY, and then apply Theorem 20 and the properties of parallelograms. What of the case when the transversals intersect on the middle parallel?

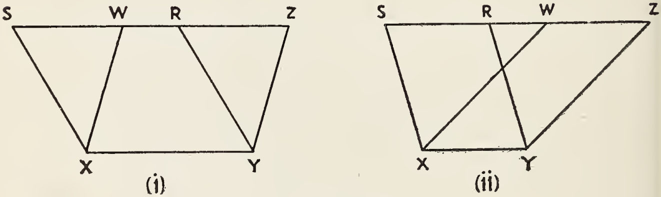
THEOREM 21. (CONVERSE OF THEOREM 20.) **The straight line joining the middle point of the two sides of a triangle is parallel to the third side and equal to one half of it.**

Hints: (a) Use a parallel to RS through T (fig. 10). Produce XY.

or (b) Assume that the line XY is not parallel to ST and use Theorem 20 and the assumption preceding Theorem 5.

V. Area

THEOREM 22. **Parallelograms on the same base and with the same altitude are equi-areal.**



(i) (ii)

Fig. 10.12

Data ||ograms XYZW and XYRS.

(1) Common base XY;

(2) Between same parallels XY, SZ.

To prove that XYZW and XYRS are equi-areal.

Hint: Show that ΔXSW ≡ ΔYRZ.

Corollary 1. **The area of a parallelogram is measured by the product of base and altitude.**

Hint: Consider the rectangle on XY between the parallels XY and SZ (fig. 12).*

Corollary 2. **Triangles on the same base and with the same altitude are equi-areal.**

Data Δs XYR and XYW.

(1) Common base XY.

(2) Between same parallels XY, RW.

Hint: Complete the ||ograms XYRS and XYWZ.

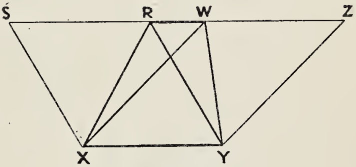


Fig. 10.13

Corollary 3. **The area of a triangle is measured by one-half the product of base and altitude.**

Hint: This follows directly from Corollary 1 and Corollary 2.

Note: This corollary may be proved as follows:

(i) Prove it for a right triangle, assuming the formula for the area of a rectangle.

(ii) Prove it for any triangle by expressing the area of the triangle as the sum or difference of the areas of two right triangles.

---

*The formula for the area of a rectangle is to be assumed.

THEOREM 23. (CONVERSE OF COROLLARY 2, THEOREM 22.) **Equi-areal triangles on the same base have the same altitude. If the triangles are on the same side of the base, they are between the same parallels.**

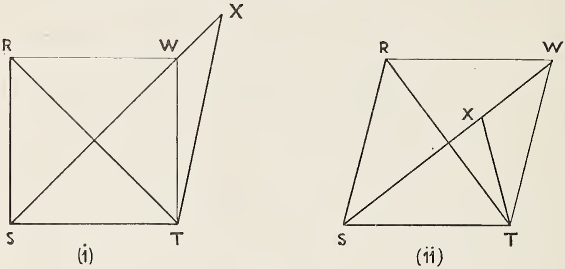


Fig. 10.14

Data Δs RST and XST.

(1) ΔRST = ΔXST.

(2) The triangles are on the same side of the common base ST.

To prove that RX is || to ST.

Construction. From R draw a line RW || to ST.
RW must either
(i) cut SX at X,
or (ii) cut SX or SX produced at a point W other than X.
Join TW.

Proof. Hint: Suppose (ii) is true and show that ΔWST = ΔXST, which is impossible. Hence (ii) is false, and (i) is true.

Note: The theorem follows at once by algebra from Corollary 2.

Construction 7. *To construct a triangle equal in area to a given quadrilateral.*

Hint: Draw a diagonal and apply Corollary 2, Theorem 22 to form the required triangle.

VI. The Side Property of a Right Triangle

THEOREM 24. **In a right triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.**

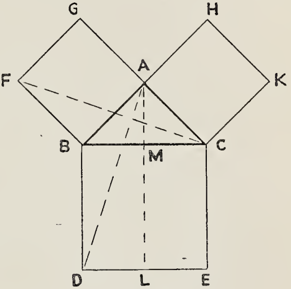


Fig. 10.15

Data ABC is a triangle, right-angled at A.

The figures BE, CH, AF are squares described upon BC, CA, AB respectively.

To prove that sq. BE = sq. CH + sq. AF.

Construction. Through A draw AL $\parallel$ to BD (or CE). Join CF, AD.

Outline of proof:

(i) Use Theorem 8 to prove that $\triangle ABD \equiv \triangle FBC$;

(ii) Use Theorem 2 to prove that CAG is a straight line;

(iii) Since CAG is $\parallel$ to BF use Theorem 22, Corollaries 1 and 3, to prove that $\triangle FBC = \frac{1}{2}$sq. AF;

(iv) Prove that $\triangle ABD = \frac{1}{2}$ rect. BL, and hence that sq. AF = rect. BL;

(v) Prove that sq. CH = rect. CL, and hence that sq. AF + sq. CH = sq. BE.

Alternative proofs are suggested by figs. 16 and 17.

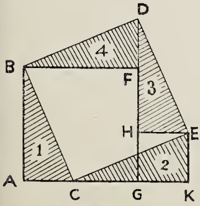
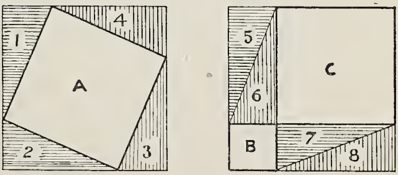


Fig. 10.16



(i) (ii)

Fig. 10.17

THEOREM 25. (CONVERSE OF THEOREM 24.) **If a triangle is such that the square on one side is equal to the sum of the squares on the other two sides, then the angle contained by these two sides is a right angle.**

Data ΔRST.
(1) $r^2 = s^2 + t^2$.

To prove that $\angle$R is a rt. $\angle$.

Construction. Construct ΔXYZ with $z = t$, $y = s$ and $\angle$X = rt. $\angle$.

Proof. Hint: Show that $x = r$ and hence ΔRST $\equiv$ ΔXYZ.

Note: An indirect proof is possible. $\angle$R is either (i) a rt. angle or it is (ii) not a rt. angle. Assume (ii) and show it to be false. Work with the side-relation, using algebraic notation (r, s, t) for the sides.

VII. Simple Loci in the Plane

THEOREM 26. **The locus of points which are equi-distant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.**

Since a locus comprises all the points which satisfy the given conditions and no other points, it is necessary to prove that (I) if a point is equidistant from the fixed points then it lies on the perpendicular bisector and conversely that (II) if a point is on the bisector it is equidistant from the fixed points.

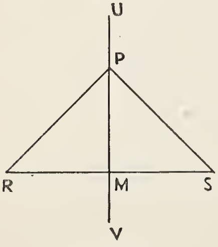


Fig. 10.18

I. *Data* R and S are the fixed points.
(1) P is equidistant from R and S, i.e., PR = PS.

To prove that (i) P lies on UV, the perpendicular bisector of RS.

Construction. Join RS.
M is the mid-point of RS.
Join MP.

Hint for proof: $\Delta RMP \equiv \Delta SMP$. (S.S.S.)

II. *Data* R and S are fixed points.

UV is the perpendicular bisector of RS.

(1) P lies on UV, ((i) I).

To prove that (i) $PR = PS$. ((1) I).

Hints for proof: (a) If $P \equiv M$, evidently $PR = PS$;

(b) If P is distinct from M prove $\Delta RMP \equiv \Delta SMP$. (S.A.S.)

THEOREM 27. **The locus of points which are equi-distant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.**

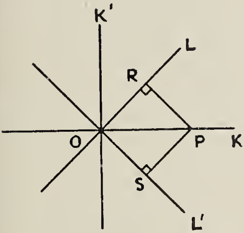


Fig. 10.19

I. *Data* L and L′ are two intersecting lines

K and K′ are the lines which bisect the angles formed by L and L′.

(1) $PR = PS$ where PR and PS are the perpendiculars from any point P on L and L′.

To prove that (i) P lies on K or K′.

Hint for proof: $\Delta OPR \equiv \Delta OPS$. (Th. 11.)

II. *Data* (1) P lies on K or K′. ((i) I)

To prove that (i) $PR = PS$. ((1) I)

Hint for proof: $\Delta OPR \equiv \Delta OPS$. (A.A.S.)

VIII. Similar Figures

A. PROPORTIONAL DIVISION

THEOREM 28. **If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.**

The proof of this theorem was given in §4.

COROLLARY. **If a straight line is drawn parallel to one side of a triangle, the two triangles so formed have their corresponding sides in proportion and their corresponding angles equal.**

Hints: In fig. 1, p. 124, draw YZ || to XS.
XYZS is a ||ogram and XY = SZ.

$$\frac{RX}{RS} = \frac{RY}{RT} = \frac{SZ}{ST}. \quad \text{Why?}$$

$$\therefore \frac{RX}{RS} = \frac{RY}{RT} = \frac{XY}{ST}.$$

THEOREM 29. (CONVERSE OF THEOREM 28.) **If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.**

Hints: (a) The method of proof suggested for Theorem 28 may be used;

(b) The method of elimination may be applied.

B. SIMILAR TRIANGLES

THEOREM 30. **If two triangles are similar, corresponding sides are proportional.**

Hint: This follows directly from the definition of similar figures since *all* corresponding lengths are in the same proportion in the two figures.

‡THEOREM 31. (CONVERSE OF THEOREM 30.) **If two triangles have their corresponding sides in proportion, the triangles are similar.**

COROLLARY. (Right triangles.) **If two right triangles have two pairs of corresponding sides proportional the ratio of the third pair of sides is equal to that of either of the other pairs, and the triangles are similar.**

Data $\Delta \mathsf{RST}$ and $\Delta \mathsf{R'S'T'}$.

$$(1)\ \frac{r}{r'} = \frac{s}{s'}\cdot$$

To prove that $\dfrac{t}{t'} = \dfrac{r}{r'} = \dfrac{s}{s'}\cdot$

Hints: There are three possibilities according as R, S, or T is a right angle, and hence $\mathsf{R'}$, $\mathsf{S'}$ or $\mathsf{T'}$ is a right angle.

In terms of the sides these possibilities are

$$\begin{array}{rll}\text{(i)} & r^2=s^2+t^2; & (r')^2=(s')^2+(t')^2,\\ \text{(ii)} & s^2=r^2+t^2; & (s')^2=(r')^2+(t')^2,\\ \text{(iii)} & t^2=r^2+s^2; & (t')^2=(r')^2+(s')^2.\end{array}$$

Let $\dfrac{r}{r'}=k$. Then $r=kr'$. Likewise $s=ks'$. (*Data*).

It is not difficult to show in (i), (ii) or (iii) that $t=kt'$.

THEOREM 32. **If two triangles are similar, corresponding vertex angles are equal, i.e., the triangles are equi-angular.**

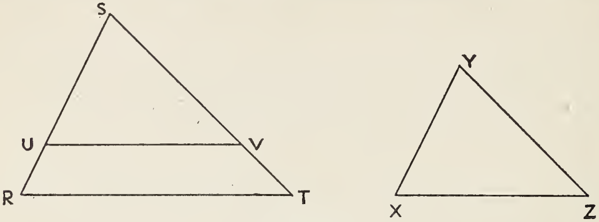


Fig. 10.20

Data ΔRST and ΔXYZ.

Since the triangles are similar:

$$\frac{RS}{XY} = \frac{ST}{YZ} = \frac{RT}{XZ}$$

Construction. Draw UV with $US = XY$, $VS = ZY$.

To prove that ΔRST and ΔXYZ are equi-angular.

Proof. Hints: ΔUSV and ΔRST have their sides proportional and are equiangular. Why?

Also $\Delta USV \equiv \Delta XYZ$. Why?

THEOREM 33. (CONVERSE OF THEOREM 32.) **If two triangles are equiangular, the triangles are similar.**

Hint: Use the construction of Theorem 32.

COROLLARY. (Right triangles.) **Two right triangles with one pair of corresponding acute angles equal are similar.**

THEOREM 34. **If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar.**

Hint: Use the construction of Theorem 32 with the equal angles at S and Y.

CHAPTER XI

THE CIRCLE. LENGTH AREA, CHORDS

*"What is the cleverest and most important thing man has invented since those days, . . ., when he first appeared and started his long climb towards civilization . . .? I wonder what most people would reply . . . Radio? Aeroplanes? The steam engine? My own answer goes much further back, and chooses a far simpler thing . . . It is the wheel."**

1. Introduction.

The Greek geometers, as you have seen, played the game of geometry at first with two rules.

This geometry of straight-edge and compasses was thus exclusively concerned with the properties of figures composed of straight lines and/or circles.

The properties of straight line figures have been explored in preceding chapters. We are now to consider the theory of the circle.

EXERCISES. Common Words

1. Explain or illustrate with figures the meaning of: (a) centre; (b) radius; (c) diameter; (d) semicircle; (e) chord; (f) arc; (g) minor and major arcs; (h) secant; (i) segment; (j) sector; (k) circumference; (l) tangent.

2. What is the definition of a circle as the locus of a point? According to this definition, is a circle a plane region or a curve? What would be meant by (i) the circumference, (ii) the area of a circle? Is the circumference a curve or a length (number and unit)? Is the area a surface?

---

*Boumphrey, *The Story of the Wheel* (Macmillan, 1932), p. 10. This short book is fascinating bit of history, with interesting accounts of wheel-making, the potter's wheel, early lathes, road-making, early stage coaches, steam coaches, early trains, etc.

2. The circumference and area of a circle.

You saw earlier that the problem of finding the area of a circle was one of the three problems not solvable by the use of ruler and compasses.

Archimedes (c. 225 B.C.) invented a spiral curve by means of which he was able to find the circumference of a circle and hence its area, for he also knew that

$$\text{Area} = \frac{\text{radius} \times \text{circumference}}{2}.$$

His results may be expressed as follows:

$$\text{Circumference} = 2\pi\ (\text{radius}),$$
$$\text{Area} = \pi\ (\text{radius})^2,$$

where in decimal notation, π is a non-repeating, non-terminating number whose value to six figures is

$$\pi = 3.14159\ldots$$

Archimedes knew that π lay between the two numbers 3 10/17 and 3 1/7.*

The formulas for area and circumference cannot be proved here. They are best obtained by the calculus methods. The calculus was invented by *Newton* and *Leibniz* (c. 1680).

Summary

Circumference $C = 2\pi r = \pi d,$

Area $A = \pi r^2.$

EXERCISES. Circumference and Area

CIRCUMFERENCE

1. Use the wheel of a bicycle to verify the formula for circumference. The circumference may be measured by rolling the wheel along a floor. Other sizes of wheels may be used, if more convenient.

---

*The common use of the approximation 22/7 for π is unfortunate. A more sensible approximation for making estimates is the simpler number 3. In any event use the decimal form of π to the required number of significant figures.

A table of various historical approximations to π is given in Sanford, *A Short History of Mathematics* (Houghton-Mifflin, 1930).

2. $\pi = 3$ is used in the Old Testament (I *Kings*, 7:23). A cubit was the length of a man's forearm from his elbow to his finger-tips. Use 18 in. as a cubit to find the error in stating that the circumference is 30 cubits for a diameter of 10 cubits.

Results in the following exercises including those on area should be given, at most, to 3 *significant figures.*

3. Calculate the circumference of a circle whose radius is (i) 14 in., (ii) 21 miles.

4. Calculate the circumference of the earth, measured round the equator, taking radius = 3,960 miles.

5. How far does a wheel roll in one revolution if its diameter is 7 in.?

6. In fig. 1, **AD** is divided into three equal parts and all the arcs are semicircles; show that the four curved lines which connect **A** with **D** are of equal length.

7. Find r, the radius, in terms of C, the circumference of a circle.

8. Calculate the radius of a circle of circumference (i) 44 ft., (ii) 60 cm.

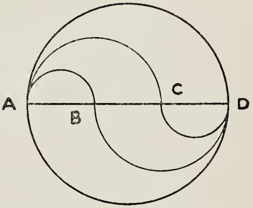


Fig. 11.1

9. What is the diameter of a wheel which makes 36 turns in running over a distance of 66 ft.?

10. What fractions of a circumference are arcs of 90°, 60°, 120°, 1°, 35°, 300°?

11. Calculate the lengths of arcs of 20° and 150° in a circle of radius 12 in.

12. The circumference of a circle is 12 in. and the length of a certain arc is 4.5 in. What decimal of the circumference is the arc? What angle does the arc subtend at the centre?

13. Calculate the distance measured along a meridian of longitude from the equator to Lat. 50° N. (take circumference of earth = 25,000 miles).

14. Calculate the length of 1° of longitude at the equator.

15. A cog-wheel with 10 teeth, driving a larger one with 50 teeth, makes one revolution. Through what angle has the larger turned?

16. A driving-wheel of diameter 14 in., containing 64 teeth, is connected by a continuous chain to a small gear-wheel containing 8 teeth. What length of chain passes round while the small wheel makes 1 revolution?

17. A beam resting across a cylindrical pipe of diameter $3\frac{1}{2}$ ft. can rock through an angle of 20° without slipping. What length of the beam comes into contact with the pipe?

18. A marble, diameter 1.5 cm., rolls down an arc of 60° of a hemispherical bowl of diameter 30 cm. How many revolutions does it make?

19. A large wheel of diameter 3 ft. 6 in. is fixed to an axle of diameter $3\frac{1}{2}$ in. round which a rope is coiled with a weight attached; the wheel and axle can turn about a line through their common centre. By turning the wheel, the weight is raised 10 in. Through what angle has the wheel turned?

Area

20. Draw a circle on squared paper (use radius 2 in., if inch paper, or 5 cm. if centimetre paper). Find the area of the circle by counting squares. Check with the formula for area. Since $\pi = A/r^2$, what is your measured value of π?

21. Calculate the area of a circle whose radius is (i) 2 in., (ii) $3\frac{1}{2}$ ft., (iii) 15 cm.

22. The radius of one circle is twice the radius of another; how many times does the area of the greater contain the area of the smaller?

23. Find the ratio of the area of a circle to the area of the circumscribing square.

24. Two concentric circles have radii R, r. What is the area between them?

25. The outer and inner diameters of a pipe are 14 in. and 12 in. What is the area of a cross-section?

26. Calculate the radius of a circle whose area is (i) 44 sq. in., (ii) 14·3 sq. cm.

27. A circular track encloses 10 acres of ground. What is the diameter of the track? (1 acre = 4,840 sq. yd.)

28. A circular disc has a circular hole drilled in it, a diameter of the hole being a radius of the disc. What fraction of the original disc is now left? Make a freehand sketch of the figure.

3. Exploratory exercises.

Draw freehand figures to answer the following questions.

1. In equal circles does an angle at the centre of 60° subtend the same arc length on each circle? The same chord length?

2. If in a given circle the central angle is doubled, will the arc subtended by the angle be doubled? Will the chord subtended by the angle be doubled?

3. In two unequal circles will a central angle of 60° subtend arcs whose lengths are proportional to the radii of the circles?

4. In a given circle is the area of a sector with a central angle of 150° double the area of a sector with angle 75°?

5. In a given circle will a sector whose area is three times that of another sector have three times the central angle?

6. Are 60° sectors of circles of various sizes similar figures?

7. In a given circle are any two arcs proportional to their central angles?

8. In a given circle are any two sector areas proportional to the sector angles?

9. In a given circle are any two chords proportional to their central angles?

10. Is the area of a sector of angle a° $\frac{a}{360}$ of the area of the circle?

4. Facts about chords.

1.—Draw a circle of radius 1.5 in. and mark the centre O. Cut out the circle. Draw a chord RS not a diameter. Fold the circle so that R falls on S; crease.

What facts do you observe about the crease line?

2.—Let M be the mid-point of RS and let XY be a line. Write the three possible theorems which have for data any two of the following statements, the remaining statement being the conclusion:

(a) XY bisects the chord RS at M;

(b) XY passes through O;

(c) XY is perpendicular to the chord RS.

Do you think these theorems are true? Can you prove them?

3.—A circle is to pass through two points R and S. On what line does the centre lie?

4.—Given △RST. The perpendicular bisectors of the sides intersect in the same point O, which is equidistant from the vertices R, S, T. Why?

5.—The point O in para. 4 is the centre of a circle which passes through R, S, T. Why?

This circle is said to *circumscribe* the triangle, and the triangle is said to be *inscribed* in the circle.

If any three points X, Y, Z not on a straight line are given, how many circles can be passed through them?

The facts suggested above are summarized in Theorems 40 and 41 at the end of the chapter. Theorem 39, which is assumed to be true, was indicated in §3, item 1. Note the form of the theorem where the truth of any one of (a), (b), (c) guarantees the truth of the other two.*

---

*Why are these theorems about circles numbered consecutively with the theorems about straight lines? Why not start afresh: Theorems 1, 2, 3, etc.?

EXERCISES. Scale Drawing

Two of the perpendicular bisectors of the sides of a triangle intersect in the centre of the circumscribing circle. Use the third bisector to check the centre in the exercises which follow.

1. Find the centre and radius of the circle circumscribing a triangle whose sides are (i) 3 in., 4 in., 4 in.; (ii) 3 in., 4 in., 5 in.; (iii) 3 in., 4 in., 6 in.

2. (Graph paper.) Find the radius and the co-ordinates of the centre of the circumscribing circle.

(a) (0,3), (2,0), (−1,0); (b) (0,1), (3,0), (−3,0);
(c) (0,2), (4,0), (−1,0).

3. The arch of a bridge is a portion of a circle. The width of the arch is 20 ft. and its height is 8 ft. Draw the arch to scale and measure its radius.

4. A window is in the shape of a rectangle surmounted by an arc of a circle. The width of the window is 6 feet. Its least height is 10 feet, and its greatest height 12 feet. Draw the outline of the upper part of the window to a scale of 1 in. to 2 ft.

5. A Dutch barn has a roof whose section is an arc of a circle. The framework supporting the roof is triangular with two beams of 15 ft. length meeting at an angle of 130°. What is the radius of the circular arc?

5. Chord length.

The perpendicular from the centre upon a chord bisects the chord; therefore the radius, distance of chord, and half-length of chord are connected by Pythagoras' theorem. (See fig. 2.)

Note also that $l = r \sin a$, and $d = r \cos a$.

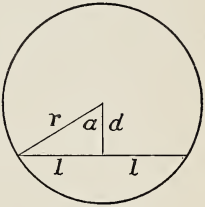


Fig. 11.2

EXERCISES. Chord Length

1. Calculate the distances from the centre of a circle, radius 5 cm., of chords whose lengths are (i) 6 cm., (ii) $2a$ cm.

2. Calculate the lengths of chords of a circle, radius 2.5 in., whose distances from the centre are (i) 1.5 in., (ii) d in.

3. Calculate the radius of a circle, given that a chord 3 in. long is 2 in. from the centre.

4. A chord CD of a circle, whose centre is O, is bisected at N by a diameter AB. $OA = OB = 5$ cm., $ON = 4$ cm. Calculate CD, CA, CB.

5. The arch of a bridge is an arc of a circle. If the span of the bridge is 50 ft. and the greatest height 10 ft., calculate the radius of the circle.

6. The lengths of two parallel chords of a circle of radius 6 cm. are 10 cm. and 6 cm. respectively. Calculate the distance between the chords. (There are two cases.)

7. Given that a chord 12 cm. long is distant 2.5 cm. from the centre, calculate (i) the length of a chord distant 5 cm. from the centre, (ii) the distance from the centre of a chord 6 cm. long.

Trigonometric Exercises

✓**8.** Compute l in fig. 2 for $r = 5$ and (i) $a = 30°$, (ii) $a = 40°$, (iii) $a = 70°$, (iv) $a = 90°$. Verify roughly with drawings.

9. A circle of radius 5 in. is given.

(a) Show that if a chord subtends an angle $2a$ at the centre, the length of the chord is $c = 10 \sin a$.

(b) Is $\sin 2a \equiv 2 \sin a$? Try this for various angles.*

(c) If angle is doubled, is subtended chord doubled? Is subtended arc?

✓**10.** Compute (i) the length of arc; (ii) the area of the sector when a, fig. 2, is (a) 30°, (b) 60°, (c) 80°.

11. (a) Show that the area, fig. 2, of the triangle with base $2l$ and altitude d is $ld = r^2 \sin a \cos a$;

(b) Show that the sector area, central angle $2a$, is $\frac{a}{180} \pi r^2$;

---

* $\equiv$ is the sign of identity. You have used it for identical (congruent) figures.

(c) Show that the segment area for chord $2l$ is

$$S=r^2\left(\frac{\pi a}{180}-\sin a\cos a\right)\cdot$$

Find this area when $a=45°$. Check by drawing on graph paper with $r=5$.

12. The contents of the tank on a railway gasoline carrier may be found by ex. 10 when the dimensions of the tank and the depth of the gasoline are known. Find S (ex. 11) when the radius of the tank is 2.5 ft. and the depth of the gasoline is (i) 1 ft., (ii) 4 ft. Then find the volume of gasoline if the tank is 25 ft. long.

6. The section of a sphere.

If a plane cuts a sphere, the section is a circle. If the cutting plane passes through the centre of the sphere, the circle is called a great circle; if the plane does not pass through the centre of the sphere, the section is a small circle. (See fig. 3.)

The foot of the perpendicular from the centre of a sphere on a small circle is the centre of the small circle.

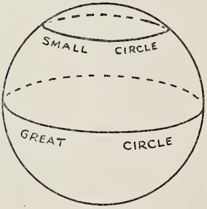


Fig. 11.3

Example

A sphere of radius 8 cm. is cut by a plane distant 5 cm. from the centre. Calculate the radius of the circular section.

(Fig. 4 shows a general view of the figure, but note that fig. 5 shows all that is essential for the calculation.)

O is the centre of the sphere, **N** the centre of the circular section, **A** a point on the circle.

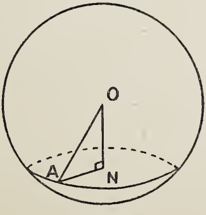


Fig. 11.4

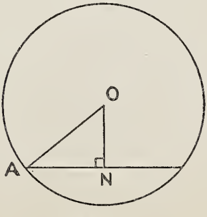


Fig. 11.5

Since $\angle ONA$ is a rt. angle,

$$OA^2 = ON^2 + NA^2.$$

$$\therefore \quad 8^2 = 5^2 - NA^2,$$

$$\therefore NA^2 = 64 - 25 = 39,$$

$$\therefore NA = \sqrt{39} = 6.24,$$

$\therefore$ the radius of the section is 6.24 cm.

EXERCISES. The Sphere

1. A wooden ball of 4 in. radius is planed down until there is a flat circular face of radius 2 in. If the ball is now made to stand on the flat face, how high will it stand?

2. The distance of the plane of the Arctic circle from the centre of the earth is 3,700 miles (to the nearest 100 miles); the radius of the earth is 4,000 miles. Find the radius of the Arctic circle.

3. A ball of radius 4 cm. floats in water immersed to the depth of $\frac{3}{4}$ of its diameter. Calculate the radius of the water-line circle.

4. A sphere of 6 in. diameter rests on the top of an open hollow cylinder whose inner diameter is 4 in. To what distance will the sphere project above the top of the cylinder? (See fig. 6.)

5. A thin rectangular plate 8 in. by 12 in. is placed in a hemispherical bowl of radius 8 in.; how far is the plate from the centre of the bowl?

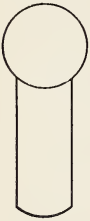


Fig. 11.6

7. The regular polygon.

A *regular polygon* is a polygon that has all its sides equal and all its angles equal.

What is the regular 3-gon (triangle)? 4-gon (quadrilateral)? Each of these two figures has a circumscribing circle and indeed so has every regular polygon. The regular polygon has a "centre" and a "radius," the centre and radius of the circumscribing circle.

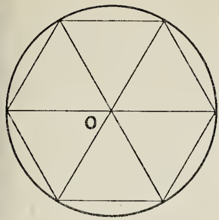


Fig. 11.7

Example 1

Inscribe a regular 6-*gon* (*hexagon*) *in a circle of radius* 2 *in.*

Since the figure is to have six sides, the 360° at O is divided into 6 equal parts of 60° each. The extremities of the radii are then joined to form the hexagon. The side of the hexagon is 2 in., and the angle is 120°.

Example 2

What is the length of side of a regular 5-*gon* (*pentagon*) *which is inscribed in a circle of radius* 2 *in.?*

Here the angle subtended by each side at the centre is $360°/5 = 72°$. The length of side is thus $2(2 \sin 36°) = 2.35$ in.

The angle of the pentagon is 108°.

A bolt head is a common example of the regular polygon and its circumcircle and forms one of the standard exercises in mechanical drawing.

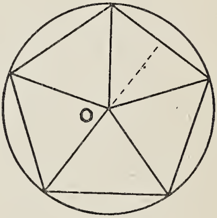


Fig. 11.8

EXERCISES. The Regular Polygon

1. Inscribe a regular 8-gon (octagon) in a circle of radius 2 in. Find the side and angle of the octagon.

2. Find the length of side of (i) a square, (ii) an equilateral triangle which can be inscribed in a circle of radius 3 in.; of radius r in.

3. *The inscribed regular polygon may be used to obtain an approximation to* π *on the lower side*, viz., find the length of side of the regular 60-gon of radius r in. and hence find the perimeter of the polygon. The result should be $6.276\,r$ in. The circumference is $2\pi r$ and so $2\pi > 6.276$, or $\pi > 3.138$.

Then re-compute for the regular 120-gon.

8. The shape of a circular arc.

All circles have the same shape since any one may be obtained from any other by enlargement or reduction. They are similar figures.

Fig. 9 shows a number of circular arcs on the same chord **RS**.

These arcs do not have the same shape, although they are all parts of circles. Can you explain why?

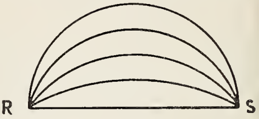


Fig. 11.9

A number of arcs on the same chord are shown in fig. 10. The centres and radii are also shown. The figure makes it clear that the more sharply the curve turns, the larger is the central angle and the smaller is the radius.

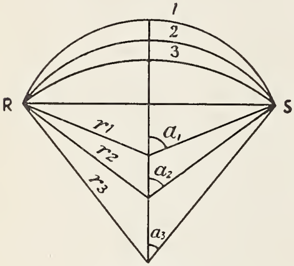
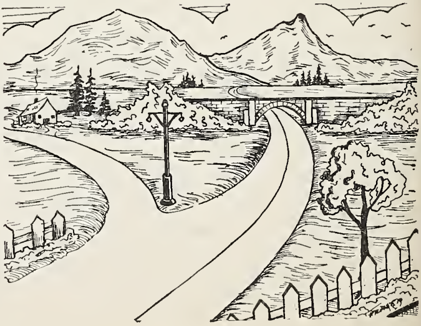


Fig. 11.10

In railroad and highway construction, circular arcs are used for turns and the central angle of an arc is used to show the variation in arc shape. The term used is *degree of curve* and the engineer speaks of a 3° curve, a 10° curve, etc.

In order to obtain this angle measure of shape, the engineer refers all arcs to the same chord length of 100 ft.



Definition: Degree of Curve. *If a circular arc on a* 100 *ft. chord subtends an angle of* **D**° *at the centre, this angle* **D**° *is spoken of as the degree of curve, and the arc is said to be a* **D**° *curve.*

On heavy-traffic, high-speed railroads, curves are usually no sharper than 5° or 6°; on light-traffic, relatively low-speed lines, curves may exceed 20°.

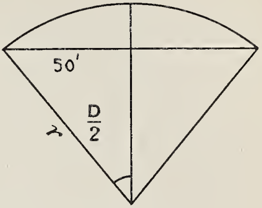


Fig. 11.11

The radius of an arc may be found when **D** is known. Fig. 11 shows that $r = 50/\sin \frac{1}{2}\mathbf{D}$. Remember that the radius is r feet.

In mathematical work the sharpness with which an arc turns is measured by $\frac{1}{r}$, since as the arc flattens, r increases. (See fig. 10.) This measure is called the *curvature* of the circle. The value $\frac{1}{r}$ of the curvature is not an arbitrary one, but measures the rate at which the curve turns away from its tangent line.

EXERCISES. The Shape of a Circular Arc*

✓**1.** Compute the radius of a 10° curve. Draw a part of this circle. To check the curve draw a 100 ft. chord on it and measure the central angle to be 10° approx.

✓**2.** Compute the radius of a (i) 5°, (ii) 15°, (iii) 20° curve. Draw an arc with a 100 ft. chord for each of these circles.

✓**3.** A highway makes a right-angle turn. The turn is designed with a (i) 4°, (ii) 8°, (iii) 16° curve. Compute the radius and draw the turn in each instance. Use the same scale for each drawing. It is assumed that each turn is a quarter circle.

4. The engineer calls the beginning of a curve the *point of curve* (P.C.), while the end of the curve is called the *point of tangent* (P.T.). He often measures the length of a curve approximately by using 100-ft. chords. Find by this method the approximate length of the curves of ex. 3, using the drawings and scaling off the distance. Also compute the chord distance from P.C. to P.T. and check with the drawings.

---

*These exercises require the use of the sine function. The results of ex. 1-2 should be kept for use in the next chapter.

9. Exercises. Chords (Proofs).

You may use Theorems 39-41 and the chord length properties of §5. The use of the trigonometric functions is nowhere necessary in these exercises.

Example

A quadrilateral ABCD *is inscribed in a circle and* AB = CD. *Prove that* AC = BD.

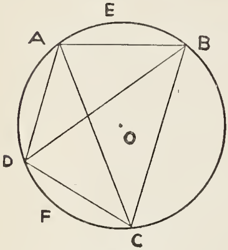


Fig. 11.12

Data Circle, centre O. A, B, C, D on the circle.
AB = CD.

To prove that AC = BD.

Discussion. (a) David gave a proof, using arc and chord properties. He argued as follows:

"If I assume that AC = BD then arc AFC = arc BED. Hence arc AB = arc DC and finally AB = CD."

"Now," said David, "the steps in my argument can be retraced to prove AC = BD." David's reasoning appears below as *Proof.*

(b) Jerry was not as fortunate as David. He said:

"I tried to prove ΔABD ≡ ΔADC for I knew that AB = CD and AD is common to both triangles, but I could not see how to obtain another pair of equal angles or sides."

"But," added Jerry, "I am sure that the triangles *are* congruent!"

Jerry was right, but before he can *prove* the triangles congruent (without using David's argument to do so), he will need to know certain facts about angles in a circle. (See chap. XIII, §2).

Proof. If AB = CD then arc AEB = arc DFC, since if chords in the same circle are equal, then corresponding arcs are equal. (Theorem 39.)*

---

*Corresponding arcs are easily specified. Arc AEB corresponds to arc DFC for E and O (centre) are on opposite sides of AB, and F and O are on opposite sides of CD. Likewise arcs AFB and CED correspond, since F and O are on the same side of AB while E and O are on the same side of CD.

Hence arc DA + arc AEB = arc AD + arc DFC,
or arc DEB = arc AFC.
∴ BD = AC,

since if arcs are equal in the same circle, then their chords are also equal. (Theorem 39.)

EXERCISES. Chords (Proofs)

1. Prove that the locus of the middle points of a set of equal chords of a circle is a concentric circle.

2. If two chords make equal angles with the diameter through their point of intersection, they are equal.

√ **3.** If a chord cuts two concentric circles in A, B; C, D, then AC = BD. (Draw the perpendicular from the centre on to the chord.)

4. If one chord of a circle is greater than another, it is nearer to the centre. (Use Pythagoras' theorem.)

5. The shortest chord that can be drawn through a point inside a circle is that which is perpendicular to the diameter through the point. (Prove that it is farthest from the centre.)

6. Two circles intersect at A, B; prove that the line of centres bisects AB (the common chord) at right angles.

√ **7.** ACB is a diameter of a circle, centre C, and AD is a chord whose mid-point is R. Join CR and BD and use this figure to prove that ∠ADB is a right angle.

8. Two equal circles, centres H and K, intersect at R and S; CRD is a chord through R parallel to HK and cutting the circles at C and D. Prove that CD = 2HK. (Draw perpendiculars to CRD from H and K.)

√ **9.** Two unequal circles intersect at G and H. A line FGK cuts the one circle at F and the other at K, while a parallel line RHS cuts the first circle at R and the second at S. Prove that FK = RS. (Draw perpendiculars to the chords from the centres of the circles.)

10. Two unequal circles intersect in A and B. The centres of the circles are X and Y, and S is the mid-point of XY. A straight line PAQ, drawn through A perpendicular to SA, meets the circles again at P and Q. Prove that PA = AQ.

11. PQ, PR are a chord and a diameter of a circle meeting at a point P on the circumference. Prove that the radius drawn parallel to PQ bisects the arc QR.

12. Find the locus of the centres of circles which make equal intercepts on two sides of a triangle.

The Circle

IX. Arcs and Chords

Terms: Centre, radius, diameter, arc, minor and major arcs, chord, secant, segment, sector, circumference. The regular polygon and the circumscribing circle.

‡THEOREM 39. **Two circles with equal radii have centres P, Q and chords RS, XY respectively. Of the statements**

(a) $\angle RPS = \angle XQY$,
(b) arc RS = arc XY,
(c) chord RS = chord XY,

if any one is true, then the other two are also true.

A similar statement applies to angles, arcs, and chords in a single circle. It is understood in (b) that arcs RS and XY are properly specified.

THEOREM 40. **RS is any chord, not a diameter of a circle with centre O, and XY is any line. Of the three statements**

(a) XY bisects RS,
(b) XY passes through O,
(c) XY is ⊥ to RS,

if any two are true, then the remaining one is also true.

COROLLARY. **The locus of the centres of circles through two given points is the perpendicular bisector of the line joining the points.**

THEOREM 41. **There is one circle, and one only, which passes through three given points not in a straight line.**

COROLLARY 1. **Two circles cannot intersect in more than two points.**

COROLLARY 2. **The perpendicular bisectors of the sides of a triangle meet in a point.**

Formulas

Circumference of a circle: $C = 2\pi r, \quad \pi = 3.14159\ldots,$
Area of a circle: $A = \pi r^2.$

Length of a sector, angle 2a: $\dfrac{\pi a}{90} r.$

Area of a sector, angle 2a: $\dfrac{\pi a}{180} r^2.$

Length of chord 2l and distance from centre d: $l^2 + d^2 = r^2.$

Length of chord, angle 2a : $2r \sin a.$

Distance from centre of chord, angle 2a : $r' \cos a.$

Area of segment, angle 2a: $r^2 \left(\dfrac{\pi a}{180} - \sin a \cos a\right), \; a < 90°.$

CHAPTER XII

THE CIRCLE. TANGENTS

1. The tangent to a curve.

When is a straight line tangent to a curve? This question has important consequences, for it has to do with measuring the rate of change of one variable quantity with respect to another. Speed, acceleration, curvature, the rate of chemical reaction, the rate of growth in an organism—these are a few examples. Since the sciences have much to do with variable quantities of every sort, the theory of rates has been extensively developed. This theory is known as the *Differential Calculus* and was originated by *Newton*, *Leibniz* and others in the seventeenth century.

It is necessary, then, to obtain a good working definition of a tangent to a curve. The following figures show various curves with straight lines which are tangent to them.

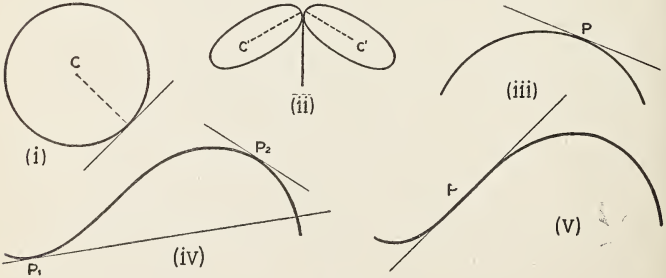


Fig. 12.1

Fig (i) shows the section of a roll of newsprint paper falling down an incline to the store-room. Fig. (ii) shows two elliptic cams in contact. Fig. (iii) represents the path of a baseball and the direction of its flight at a particular point. Figs. (iv) and (v) show an automobile fender with tangents drawn at various points. In fig. (v) the tangent is at the place where the fender changes its "bend"; on the one side it is "bent up"; on the other it is "bent down".

In view of these examples how would you define a tangent to a curve?

Here are some attempts:

1. It is a straight line which touches a curve at a point.
2. It is a straight line which, however far it may be produced, has one and only one point in common with the curve.
3. It is a line drawn perpendicular to the radius joining the centre to the point on the curve.

EXERCISE

Complete the table. Write *yes* if the definition is correct for the figure, *no* if it is incorrect, and *?* if you are in doubt.

A satisfactory definition should surely cover all the cases suggested by the above figures. You will have seen that

Figure	*Def. 1*	*Def. 2*	*Def. 3*
Fig. (i)			
Fig. (ii)			
Fig. (iii)			
Fig. (iv) (a) P_1 (b) P_2			
Fig. (v)			

definition 3 is useless, since few curves have centres and even for curves with centres (fig. (ii)) the definition is incorrect. Definition 2 comes to grief in fig. (iv), while definition 1 seems to require adjustment for fig. (v).

A satisfactory definition may be had by the use of a different idea.*

Let **P** be the point on the curve at which the tangent is required. Let any line be drawn through **P** to intersect the curve in another point **Q**. Fig. 2 shows this for a circle, with three positions of **Q**.

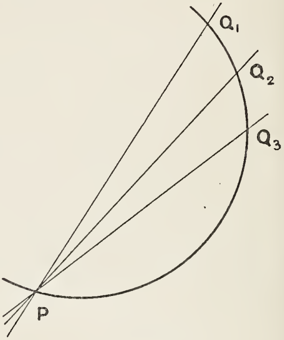


Fig. 12.2

Whenever **Q** is specified the secant line through **P** and **Q** is also specified. Now imagine **Q** to move along the curve, with **P** fixed. The secant line through **P** and **Q** pivots about **P**†. In fig. 2 think of a hoop with a stick laid across it, pivoting at **P**.

As **Q** approaches **P** closely on either side of **P** (fig. 3) the pivoting secant line, shown dotted in the figure, moves towards the tangent to the curve at **P**.

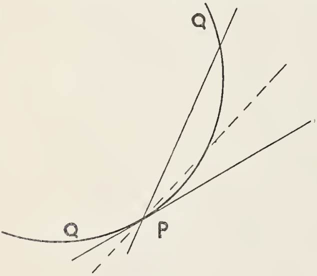


Fig. 12.3

Definition. *The tangent to a curve at a point on the curve. Given a curve and a point* **P** *on it. Let* **Q** *be any other point on the curve. The tangent to the curve at* **P** *is the limiting position of the secant line through* **P** *and* **Q** *as* **Q** *approaches* **P** *along the curve.*

---

*The locus idea of a moving point. This is "movie" mathematics, which is the basis of much modern mathematical thinking.

†It is important to visualise the entire line through **P** and **Q**; we are not considering the chord **PQ**.

P is said to be the *point of contact* of the tangent. Fig. 1 (v) and fig. 4 indicate that a tangent may cross the curve at its point of contact.

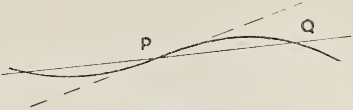
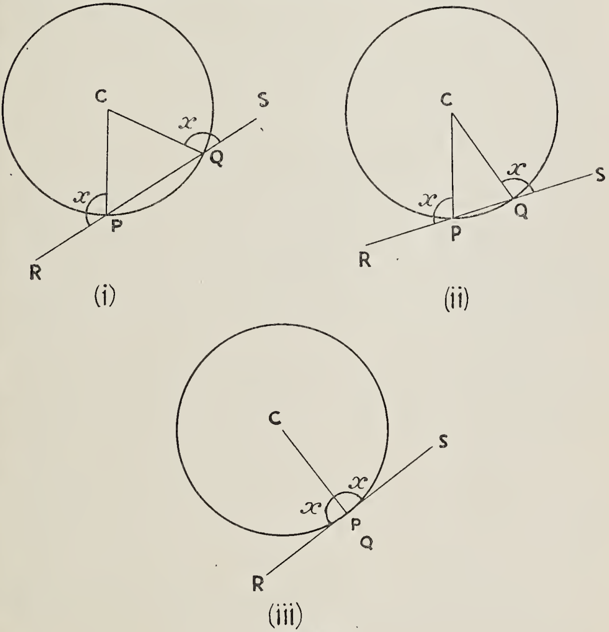


Fig. 12.4

2. The tangent to a circle.

How is the definition of §1 applied to a circle?



(i) (ii) (iii)

Fig. 12.5

Figure 5 shows the pivoting of PQ about P. Why is $\angle$ CPR = $\angle$ CQS in fig. 1? As Q approaches P, what happens to $\angle x$? State this result as a theorem about the tangent at any point of a circle. Can you state a converse theorem? Do you think the converse true? Are these theorems valid for the curve in fig. 1 (ii), p. 162?

Use a ruler to draw a tangent to a circle from a point outside the circle. It is not possible to distinguish the point of contact accurately without further construction. How would you find this point?

How many tangents are there from a point outside a circle? What can you say about them?

The facts indicated by the above questions are stated in theorem form at the end of the chapter, Theorems 42-44. These statements should be compared with your answers to the above questions. Then you should prove the theorems; they are not difficult.

The following example shows the proof of a proposition concerning tangents to circles (see ex. 19-26).

Example

AB *is a chord of a circle whose centre is* O, *and* P *is a point on the circle, with the tangent at* P *parallel to* AB. *If the tangents at* A *and* P *intersect at* T, *prove that the angles* POA *and* TAB *are equal to one another.*

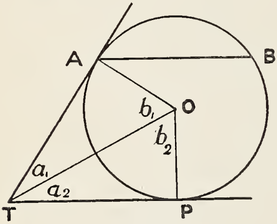


Fig. 12.6

Data Circle, centre O.

(1) TP is a tangent at P,

(2) TA is a tangent at A,

(3) AB is || to TP.

To prove that ∠POA = ∠TAB.

Discussion. ∠TAB and ∠ATP are supplementary. Hence ∠POA = ∠TAB if it can be shown that ∠ATP and ∠POA are supplementary.

Construction. Join OT, OA, OP.

Proof. ∠TAO = ∠TPO = 90° (A tangent is ⊥ to the radius through the point of contact.) (Theorem 42.)

$$\therefore \quad \angle a_1 + \angle b_1 = 90^\circ,$$
$$\text{and} \quad \angle a_2 + \angle b_2 = 90^\circ.$$

These equations may be added to find

$$(\angle a_1 + \angle a_2) + (\angle b_1 + \angle b_2) = 180^\circ,$$
$$\text{or} \quad \angle \mathsf{ATP} + \angle \mathsf{POA} = 180^\circ.$$

But since **AB** is || to **TP**, (Data (3))
then $\angle \mathsf{ATP} + \angle \mathsf{TAB} = 180^\circ$. (Int. angles.) (Theorem 5)

$\therefore$ by subtraction of the last two equations,

$$\angle \mathsf{POA} = \angle \mathsf{TAB}.$$

EXERCISES. Tangents

✓ **1.** Calculate the lengths of the tangents to a circle of radius r from a point d from the centre when (i) $r=3$ cm., $d=4$ cm., (ii) $r=2$ in., $d=4$ in., (iii) r and d are any lengths.

✓ **2.** At a point **A** of a circle (radius r, centre **O**) is drawn a tangent **AP** of length l; find **OP**.

✓ **3.** At a point **P** on the circumference of a circle of radius 4 cm. is drawn a tangent **PT** 3 cm. in length. Find the locus of **T** as **P** moves around the circle.

✓ **4.** Two circles, of radii 2 in. and 1 in., are concentric. Calculate the length of a chord of the outer circle which touches the inner circle.

✓ **5.** Show that all chords 3 in. long of a circle of radius 2 in. touch a certain concentric circle; find its radius.

6. A cylindrical boiler of diameter 22 in. stands upright in a corner between two walls. The walls are at right angles and the boiler is 4 in. from each wall. How far is the central axis of the boiler from the corner?

Exercises 7 *and* 8 *concern the length of pulley belts.*

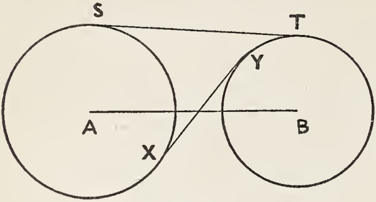


Fig. 12.7

7. Make a freehand copy of fig. 7 omitting **XY**. **A** and **B** are the centres of the circles, and **ST** is an exterior common tangent. If **AS**=6 cm., **BT**=1 cm., **AB**=13 cm., calculate the length of **ST**. (From **B** draw a line parallel to **TS**.)

8. Make a freehand copy of fig. 7, omitting **ST**. Calculate the length of the interior common tangent **XY**, if the length of **AB** and the radii are as in ex. 7.

(From **B** draw a line parallel to **YX**.)

9. Two roads each 30 ft. wide, meet at right angles. It is desired to round off the corner with a circular turn as suggested in fig. 8. **R** is mid-way between the corners **P** and **Q**. What is the radius of the turn? (Use the theorem of Pythagoras.) Draw the roadway to scale. The roadway is to be of constant width 30 ft.

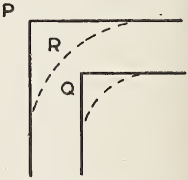


Fig. 12.8

Three Dimensions

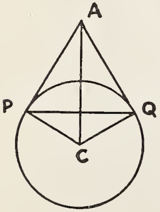


Fig. 12.9

10. If fig. 9 were spun about **CA**, what figure would be generated (i) by the circle, (ii) by **AP**, (iii) by **PQ**? Hence find the locus of the points of contacts of tangents from a fixed point to a fixed sphere.

11. What is the distance of the horizon (i.e., the length of the tangent line) from a balloon 5 miles above the surface of the earth, the radius of the earth being taken as 4,000 miles?

12. If P is a point 25 cm. from the centre of a sphere of radius 20 cm., what is the length of a tangent line from P to the sphere? Are all the tangent lines from P equal?

✓ **13.** A sphere radius 5 cm. just fits into a hollow cone of slant side 8 cm. touching around the edge of the cone. What is the distance of the centre of the sphere from the vertex of the cone?

Examples 14 *to* 18 *require trigonometry.*

✓ **14.** XY is a chord 7 cm. long in a circle of radius 5 cm. Compute the angle between the tangents at X and Y. Check with a drawing.

15. In ex. 11 what is angle of dip from the balloon to the horizon?

16. (a) What angles does ST make with AB in fig. 7?

(b) At what angle does XY cross AB in fig. 7?

17. Two railway location lines meet at an angle of 130° as in fig. 10.

(a) What is the locus of the centres of arcs of circles which touch both these lines?

(b) Draw a number of these arcs.

(c) Express the distance from V (the surveyor calls this the vertex) to the centre of the arc in terms of the radius of the arc.

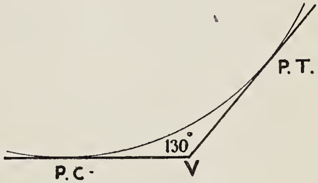


Fig. 12.10

18. Fit a (i) 10°, (ii) 20° curve to the turn in ex. 17. Draw each figure to scale. (See ex. 1-2, §8, chap. XI.)

Constructions and Proofs

19. Show how to draw a tangent to a given circle (i) parallel to a given line, (ii) perpendicular to a given line, (iii) making a given angle with a given line.

20. Show how to draw two tangents to a circle inclined to one another at a given angle.

21. AOB is a diameter of a circle whose centre is O. At C on the circumference a tangent CE is drawn. OE, parallel to AC, meets this tangent at E. Prove that EB touches the circle.

22. A straight line PQ is drawn through a point P on the circumference of a circle, making a constant angle with the tangent at P. Prove that, as P moves round the circumference, PQ touches a fixed circle.

23. Two parallel tangents meet a third tangent at U, V; prove that UV subtends a right angle at the centre.

24. The angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral are supplementary.

25. Draw a circle and a triangle ABC, the side BC touching the circle at D, CA touching at E, and AB touching at F. Mark on your figure all pairs of equal lines and prove that $AF+BC=\frac{1}{2}$ (perimeter of triangle).

26. PQRS is a quadrilateral circumscribed about a circle. Prove that $PQ+RS=QR+SP$. (Give symbols to all pairs of equal lines.)

3. The inscribed circle.

If a circle touches all the sides of a polygon, the circle is said to be inscribed in the polygon; and the polygon is said to be circumscribed about the circle (fig. 11).

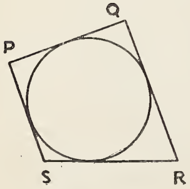


Fig. 12.11

(1) What is the locus of the centres of circles touching two lines which intersect? Draw a number of such circles touching the sides of an angle of 30°; 70°.

(2) How can you find the centre of a circle inscribed in a triangle? How many such circles are there? Why?

(3) Draw freehand a regular hexagon about a circle so that the circle is inscribed

in the hexagon. What do you notice about the points of contact? Given a circle, how would you construct to scale (i) a regular hexagon; (ii) a regular pentagon which circumscribes the circle?

EXERCISES. The Inscribed Circle

1. Draw the inscribed circle of a triangle whose sides are (i) 3, 4, 5 in., (ii) 10, 6, 10 cm. Measure the radii of the circles.

2. Draw any triangle **ABC** and produce the sides. By bisecting the exterior angles, draw a circle to touch **BC** and the other two sides produced (called an escribed circle). (See fig. 12.)

3. Draw any triangle **ABC** and the escribed circles opposite **B** and **C**.

4. A point is to be found equidistant from the three edges of a triangular field whose sides are 150 yd., 230 yd. and 215 yd. How far is the point from each side?*

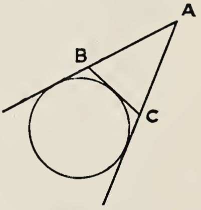


Fig. 12.12

5. What is the radius of the largest circle that can be cut out of a triangular piece of cardboard of sides 12, 10, 16 inches?

6. Draw an equilateral triangle whose sides touch a circle of radius 4 cm.

7. The top of a bolt head shows a circle inside a square, with 1/16 in. clearance. The circle is $\frac{3}{4}$ in. diameter. Draw the figure to double scale.

8. As in ex. 7 for a regular hexagon about the circle.

9. When you look on the top of an ink bottle sold by a well-known firm you see three concentric circles, the inside one being $1\frac{1}{4}$ in. diameter and the others spaced outwards at 1/8 in. intervals. A regular hexagon is inscribed in the inside circle. A regular 12-gon is circumscribed about the next circle and a regular 24-gon is inscribed in the third circle. Finally a regular hexagon is circumscribed about the third circle. Draw such a figure to double scale.

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*Find by drawing to scale. The trigonometric solution is ingenious. Can you see how to solve the problem this way?

10. A sphere rests on a triangular framework whose sides are 8 cm., 9.5 cm., and 7.3 cm. Find the radius of the circular section of the sphere by the plane of the triangle.

11. A cup in the form of a hollow right circular cone with its axis vertical is just full of water; its height is 8 in. and its slant side is 10 in. A ball is lowered slowly into the cup, of such a size that when it rests on the sides of the cup it is just covered with water. Find the radius of this ball.

Constructions and Proofs

12. In any isosceles triangle draw two equal circles touching each other and each touching the base and a side.

13. Inscribe a circle in a sector of a circle whose angle is a right angle (such a sector is called a *quadrant*).

14. Two parallel lines are cut by a third line. Draw all the circles touching the three lines.

15. Prove that if the sides of a parallelogram touch a circle the four sides of the parallelogram are equal.

16. Prove that the side of an equilateral triangle circumscribed about a circle is twice the size of an inscribed equilateral triangle.

17. A is a point outside a circle, of centre O. With centre O and radius OA describe a circle. Let OA cut the smaller circle at B. Draw BC perpendicular to OB, cutting the larger circle in P, Q. Let OP, OQ cut the smaller circle in S, T. Prove that AS, AT are tangents to the smaller circle. (This is Euclid's construction for tangents from an external point. A simpler construction will be given in the next chapter.)

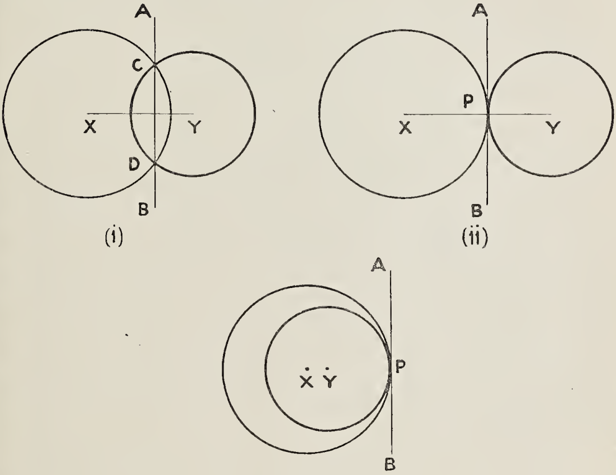
18. If a sphere exists which touches all the edges of a triangular pyramid, prove that the three sums of opposite edges are equal.

19. TH, TK are two fixed tangents to a circle centre C. A variable tangent cuts TH, TK in X, Y. Prove that XY subtends a constant angle at C.

20. The sides of a quadrilateral ABCD touch a circle. AD and BC are produced to meet in X, and BA and CD are produced to meet in Y; prove that the difference between AY and CX is equal to the difference between AX and CY.

4. Circles in contact.

If two circles touch the same line at the same point, they are said to touch one another.



(i) (ii) (iii)

Fig. 12.13

Fig. 13(i) shows two intersecting circles, their common chord produced and their centres X, Y. The whole figure is symmetrical about XY; hence XY bisects CD.

Let the small circle move to the right until the position of external contact is reached (ii). The produced chord becomes the common tangent AB, the points C and D coincide at P; and XY which formerly bisected CD now passes through P.

Now let fig. (i) change by moving the small circle to the left, until the position of internal contact is reached (iii). By similar reasoning we see that XY produced passes through P.

State your findings in the form of a theorem concerning two circles which touch one another. Can you prove the theorem? Add to the theorem a second part dealing with distance between centres.

Original Design Showing Circles in Contact

EXERCISES. Contact

✓ **1.** What is the distance between the centres of two circles of radii 15 in. and 14 in. (i) if they have external contact, (ii) if they have internal contact?

2. Draw three circles having for centres the vertices of an equilateral triangle of side 2 in., so that each circle may touch the two others externally.

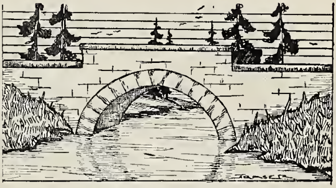
✓ **3.** Three circles, of radii 1, 2, 3 in., touch externally, each circle touching the other two. What are the distances between the centres? Draw the circles.

4. Two cylindrical pipes of radii 2.5 in. and 1.5 in. lie inside another pipe of radius 4.5 in., and touching each other. What is the distance between the centres of the pipes and the distance of the centre of the large pipe from the line joining the centres of the other two?

✓**5.** Two cylinders of diameters 10 cm. and 5 cm. have their axes parallel in the same horizontal plane and 15 cm. apart. A sphere of diameter 20 cm. rests on the cylinders. Find the height of the centre of the sphere above the plane of the axes.

6. Two circles of radii 3 cm. and 2 cm. have their centres 6 cm. apart. Find all the positions of the centre of a circle of radius 7 cm. which touches both circles in all possible ways.

✓**7.** Inscribe a circle of radius 1 in. in a semicircle of radius 2.5 in.



The simple circular arch as illustrated above in the drawing of the stone bridge is often modified by use of two or more arcs, as shown in the exercises which follow.

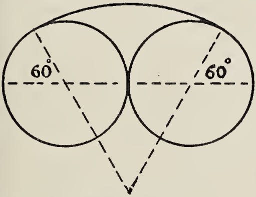


Fig. 12.14

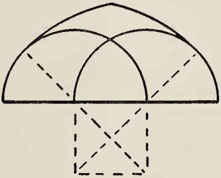


Fig. 12.15

8. *Three-centred arch.* The arch is 5 ft. wide. Copy fig. 14 to scale. What is the radius of the outside circle?

✓**9.** *Four-centred or Tudor arch.* The arch is 6 ft. wide. Copy fig. 15. The inside arcs are semicircles. What is the radius of the larger circles?

10. *Trefoil.* The radius of the outside circle is 1.5 in. and the radius of the inside circles is 1 in. Copy fig. 16 to scale.

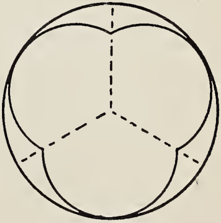


Fig. 12.16

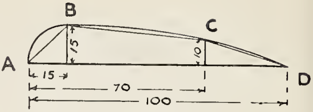


Fig. 12.17

11. *Aerofoil.* The section of an aeroplane wing is shown in fig. 17. There are three circular arcs **AB**, **BC**, and **CD** and the tangent at **A** is perpendicular to **AD**. The arcs are in contact at **B** and **C**. The numbers show lengths on the basis of **AD** = 100. Copy fig. 15 to scale. (Hint: Make use of the chords. Note the large radii for the arcs **BC** and **CD**.)

12. The railway track shown in fig. 18 consists of two straight parallel pieces **AB**, **DE**, and two circular arcs **BC**, **CD**, whose radii are 700 yards and 1400 yards respectively. Find the length of track from **B** to **D**. Draw the figure to a scale of 1 in. to 500 yards. What is the distance between the tracks?*

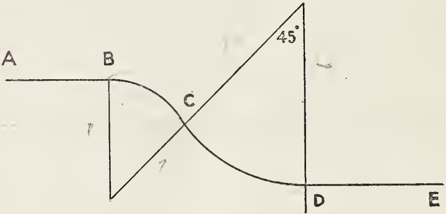


Fig. 12.18

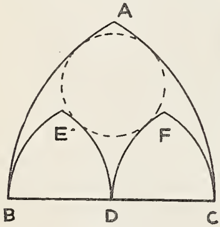


Fig. 12.19

13. In fig. 19, **BAC** is an equilateral arch, **B** being the centre of the arc **AC**, and **C** the centre of arc **BA**; **BED**, **CFD** are similar arches, **B** being the centre of **DE** and **C** the centre of **DF**, and **D** of **BE** and **CF**. What is the locus of the centres of circles touching (i) arcs **AB** and **AC**, (ii) arcs **DE** and **DF**, (iii) arcs **BA** and **DF**?

Hence explain how to construct with

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*When the distance between the tracks is given together with the points **B** and **C**, the arcs **BC** and **CD** are then determined. The centre and radius of **BC** may be found at once. To find the radius of **CD** and the centre is a good problem in trigonometry.

your instruments a circle (shown dotted in the figure) which will touch the arcs **BA**, **AC**, **DE**, **DF**.

Draw the figure carefully, taking **BC** 5 inches long.

14. Fig. 20 represents a doorway over which are a circular and two semicircular windows, the curve **ACB** being also a semicircle. If the radius of each of the two smaller semicircles is R inches and the radius of the circular window r inches, find by calculation the ratio of r to R.

Taking **AB** as 50 inches, make a drawing to scale of the part above **AB**, representing 10 inches by 1 inch. Give the radius of the circular window, and state the construction and the geometrical principles assumed.

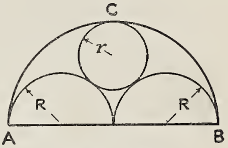

Fig. 12.20

X. Tangents

Terms: Tangent to a curve at a point.

THEOREM 42. **The tangent at any point of a circle is perpendicular to the radius through the point.**

THEOREM 43. (CONVERSE OF THEOREM 42.) **The perpendicular to the radius at a point on a circle is the tangent to the circle at the point.**

Theorem 43 gives the construction for a tangent to a circle at a given point on the circle.

THEOREM 44. **The two tangents which can be drawn to a circle from an outside point P are (i) equal in length, (ii) equally inclined to CP the join of P to the centre, and (iii) CP bisects the chord of contact RS at right angles.**

Constructions

8. *To draw a tangent to a circle at a point on a circle.*

9. *To draw the tangents to a circle from a point outside a circle.* (Ex. 17, §3.)

10. *To inscribe a circle in a given triangle.*

11. *To circumscribe a regular polygon about a circle.*

THEOREM 45. **If two circles touch, the point of contact lies on the straight line through the centres. The distance between the centres is (i) the sum of the radii when the contact is external, (ii) the difference of the radii when the contact is internal.**

CHAPTER XIII

THE CIRCLE. ANGLE PROPERTIES

1. The angle in a semicircle.

(1) Draw a semicircle on diameter **AB** with centre **C** and let **P** be any point on the semicircle other than **A** or **B**. Measure ∠**APB**. Do this for several positions of **P**. What do you find?

(2) Join **CP** (fig. 1). What sort of triangles are **ACP** and **BCP**? Prove that ∠**APB** = 90°. State this result as a theorem about a semicircle.

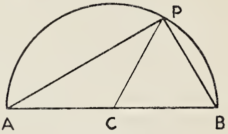

Fig. 13.1

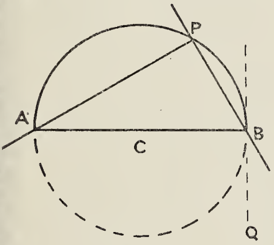

Fig. 13.2

(3) Produce **PA** and **PB** as shown in fig. 2. Let **P** move along the curve towards **B**. What can you say about ∠**APB**? About **PB**? What becomes of ∠**APB** when **P** passes through **B** on to the lower semicircle?

(4) Mark two points **A** and **B** on paper 2 in. apart. Place a set-square on the sheet so that the sides containing the right angle pass through **A** and **B**, as in fig. 3. Use successive positions of the set-square to plot the locus of the vertex of the right angle. What is the locus? In view of (3) you may wish to include **A** and **B** in the locus.*

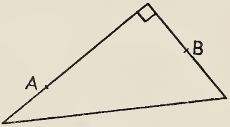

Fig. 13.3

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*Whether **A** and **B** are to be included as part of the locus is a matter for discussion and agreement. Strictly, they may be excluded, and the locus has to be stated as "a semicircle on **AB** as diameter excluding the ends of the diameter **A** and **B**." However, (3) makes the inclusion of **A** and **B** reasonable.

(5) Let **RST** be a right triangle with ∠**T** = 90° as in fig. 4. Can you prove that **T** lies on a semicircle of diameter **RS**?

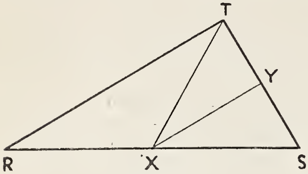


Fig. 13.4

Hints: (a) If **R**, **S**, **T** lie on a semicircle, what must be true?

(b) **X** is the mid-point of **RS**. Join **TX** and draw **XY** parallel to **RT**.

State your finding as a theorem. How is this theorem related to that of (2)? State both theorems together in locus terms.

EXERCISES. The Semicircle

✓1. Construct a right triangle with hypotenuse as base having
(a) base 4 in., altitude 1 in.
(b) base 3 in., altitude 0.5 in.
(c) base 3 in., side 2 in.

2. To construct the tangents to a given circle of centre **C** from an external point **Q**, draw a circle on **CQ** as diameter, and join **Q** to each of the intersections of the two circles. Why is this construction correct?

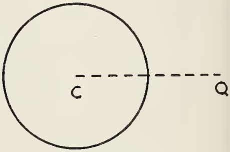
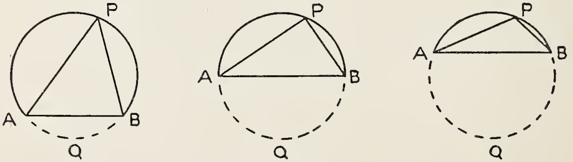


Fig. 13.5

2. The angle in a segment.



(i) Major Segment (ii) Semi · Circle (iii) Minor Segment

Fig. 13.6

Segments are of three kinds, as shown in fig. 6. The chord **AB** is the *base of the segment* while the arc **APB** is the *arc of the segment.* An angle **APB** for which **A** and **B** are the ends of the base and **P** is any point on the arc, is said to be an angle in the segment subtended by the base, briefly, an *angle in the segment.* ∠**APB** is also spoken of as an angle at the circumference subtended by the arc **AQB**. The two segments **APB** and **AQB** which together form a circle are said to be *alternate segments.*

You noted in §1 that the angle in a semicircle is a right angle, i.e., ∠**APB** $=90°$ in fig. 6(ii) for any position **P** on the arc **APB**. What of ∠**APB** in figs. (i) and (iii)?

(1) Draw a major segment of a circle on a base of 2 in. Measure the sizes of various angles in this segment. Then draw a minor segment on a base of 2 in. and measure various angles in this segment. What do you find? Compare with the angle in a semicircle. State your findings in theorem form.

(2) In order to *prove* the theorem in (1) for a major segment it is necessary to consider the angle at the centre of the circle, ∠**ACB** (fig. 7). Join **PC** and produce to **Q**. What can you say about the angles a_1 and a_2? b_1 and b_2? x, a_1, a_2? y, b_1, b_2? $(x+y)$ and (a_1+b_1)? Is ∠**ACB** constant for the segment on **AB**? What can you say about ∠**APB**? How is it related to ∠**ACB**?

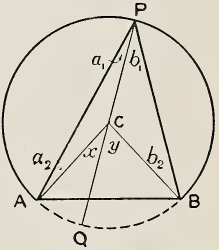


Fig. 13.7

(3) As **P** moves along the arc of the segment in fig. 7, study the behaviour of the lines in the figure. Draw the figure for each of the six positions of **P** shown in fig. 8. Does the proof in (2) require to be modified in any one of these instances?

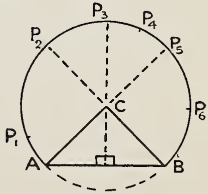


Fig. 13.8

State your findings for (2) in theorem form.

(4) What can you say of the *size* of $\angle$**ACB** at the centre of the circle (fig. 7)? Is this true for any major segment? What of the size of $\angle$**APB** in a major segment? See (1).

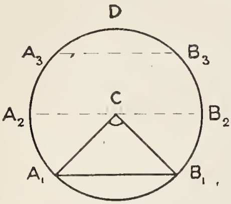


Fig. 13.9

(5) Fig. 9 shows a "movie" of a chord **AB** as it sweeps up across the circle parallel to itself. You are to think of $\angle$**ACB** freely hinged at **C**. In the first position A_1B_1, the segment A_1DB_1 is a major segment. As **AB** moves upward what can you say about $\angle$**ACB**?

(6) An angle such as is shown in fig. 10 is called a *reflex angle.* You can follow the suggestions of (2), (3), (4) to obtain theorems for a minor segment as long as you keep clearly in mind the nature of $\angle$**ACB**.

Is it necessary in the statements of the theorems to distinguish major and minor segments?

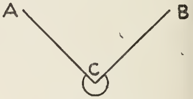


Fig. 13.10

(7) You will have noted that the theorem of §1 concerning the angle in a semicircle is a special case of the theorem concerning the angle in any segment. We say that the work of §2 is a *generalization* of that of §1.

Another instance of this idea is the generalization of the theorem of Pythagoras to give the so-called *law of cosines* for any triangle.

This law concerns the sides of a triangle and any one of its angles, for example the sides a, b, c of Δ**ABC** and the angle **C**. The law is as follows:

$$c^2 = a^2 + b^2 - 2ab \cos \mathbf{C}.$$

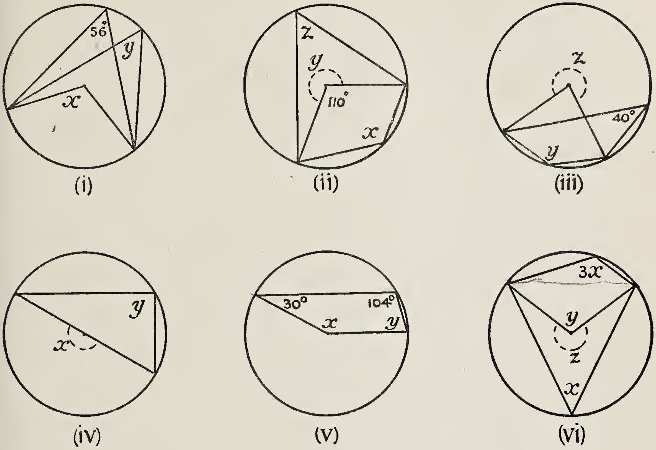
In the special case when $\mathbf{C} = 90°$, the triangle is right-angled, $\cos \mathbf{C} = 0$, and the law becomes

$$c^2 = a^2 + b^2,$$

which is the familiar theorem of Pythagoras. The law of Pythagoras is thus a special case of the more general law of cosines.

EXERCISES. The Angle in a Segment

1. Find the values of x, y, z in each of (i)-(vi), the centre of the circle being indicated by a dot.



(i) (ii) (iii)

(iv) (v) (vi)

Fig. 13.11

2. AC, BD are perpendicular diameters of a circle, P is a point on the minor arc AB; prove that the arcs AD, DC, CB each subtend 45° at P.

3. The diagonals of a cyclic quadrilateral* PQRS intersect at X, $\angle$SPR$=54°$, $\angle$RPQ$=33°$, $\angle$PSQ$=20°$. Find all the angles of the figure and show them in a freehand sketch. Explain briefly.

4. Draw a circle and a diameter AB; mark two points C, D on the circumference on opposite sides of AB; join CD, AD, AC,

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*A cyclic quadrilateral is one whose vertices lie on a circle.

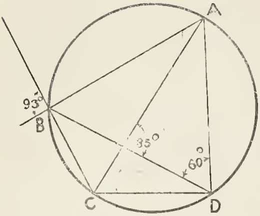

Fig. 13.12

CB, **BD**. Assuming that ∠ **BAC** = 30° and ∠ **BAD** = 40°, determine all the angles in the figure. Give your reasons carefully.

5. From the given angles, find all the angles in fig. 12.

6. Repeat ex. 5, taking ∠ **B** = 71°, ∠ **BCO** = 36°, ∠ **AOD** = 108°. Prove that in this case **AD** is || to **BC**.

3. The angle between a tangent and a chord.

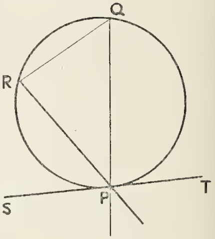

Fig. 13.13

(1) Fig. 13 shows a circle with a tangent at **P** and any chord **RP** through **P**.

Consider the major segment and any point **Q** on its arc. Join **QR** and **QP** and produce.

As **Q** moves along the arc towards **P**, what can you say about ∠ **RQP**? About **RQ**? About **QP**? What is the limiting position of ∠ **RQP**?

(2) Repeat (1) for **Q** on the arc of the minor (alternate) segment with chord **RP**.

(3) State your findings in (1) and (2) as a theorem.

(4) What can you say about the angles in alternate segments in view of the theorem in (3)? State this result as a theorem.

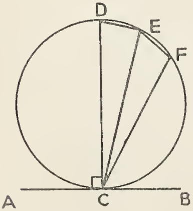

Fig. 13.14

EXERCISES. The Angle Between a Tangent and a Chord

1. In fig. 14, **AB** is tangent to the circle and **DC** is the diameter ⊥ to **AB**. **EC** and **FC** are chords. Find all the angles in the figure supposing that ∠ **BCE** = 60° and ∠ **FCE** = 20°. What angles do the chords **ED**, **CE**, **FC** subtend at the centre?

2. Find all the angles of (1) fig. 15, (2) fig. 16.

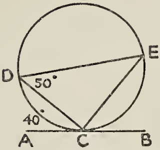

Fig. 13.15

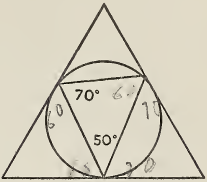

Fig. 13.16

3. If in fig. 14 arc CD = 2 arc DE, what is ∠BCD?

4. If in fig. 14 arc ED were $\frac{1}{4}$ arc DC, what would be the magnitude of ∠BCD?

5. Make a sketch of fig. 17, and fill in the number of degrees in each angle; AB is a tangent to the circle. Explain briefly the principles you use.

6. In fig. 18, AB is the tangent at A and DE is parallel to AB. Reason to find the magnitude of the angle ECA.

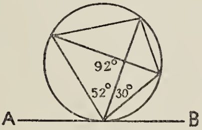

Fig. 13.17

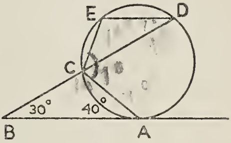

Fig. 13.18

7. ACB is a triangle inscribed in a circle. If the ∠ACB = 55°, at what angle do the tangents at A and B intersect? If ∠ACB = x°, what is the angle?

Proofs and Constructions

8. A chord AB of a circle is produced to meet the tangent at a point T in P. Show that the triangles PTA and PBT are equiangular.

9. In fig. 14 if **CF** is drawn to bisect $\angle \mathbf{BCD}$, prove that the arcs **CF** and **FD** are equal.

10. **ABC** is a triangle inscribed in a circle with $\angle \mathbf{B} = 2\angle \mathbf{A}$. Prove that the tangent at **C** is parallel to the bisector of the $\angle \mathbf{B}$.

11. Show how *to inscribe in a given circle a triangle of given angles.* (Suppose the figure to be drawn. Draw the tangent at one vertex also; then the three angles at this point are known.)

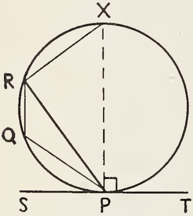

Fig. 13.19

12. You have seen how to prove theorem 52 (see end of chapter) from the definition of the tangent to a circle. Fig. 19 suggests a proof using the fact that the diameter **XP** is ⊥ to the tangent **ST**. **Q** is taken in the segment alternate to the segment **RXP**. $\angle \mathbf{PRX} = 90° = \angle \mathbf{SPX}$. Why? Hence prove $\angle \mathbf{SPR} = \angle \mathbf{RXP}$, to obtain the theorem.

4. The layout of a railway or highway curve.

The angle property of §3 is the basis of railway curve layout. The track is located at 100 ft. intervals. The point of curve (P.C.) may not coincide with the beginning of such an interval, so that the location of positions on a curve at unequal (chord) distances is often necessary.

The degree of the curve to be used is known and hence the radius of the circle and the central angle corresponding to any chord length may be computed.

The track begins to curve at **A** (P.C.) and stakes are to be driven at **B** and **C** corresponding to given but unequal (chord) distances along the track. See fig. 20. The centre of the curve is **O** and $\angle \mathbf{AOB} = x$ and $\angle \mathbf{BOC} = y$ may be computed.

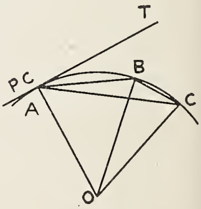

Fig. 13.20

Hence $\angle \mathbf{TAB} = \frac{x}{2}$ and $\angle \mathbf{BAC} = \frac{y}{2}$. Why?

A transit is set up at A and the position of B is located by

(i) turning off $\angle TAB = \frac{x}{2}$ from the track line AT,

(ii) measuring the given distance AB along this line of sight AB.

Likewise $\angle BAC = \frac{y}{2}$ may be turned off to give the line of sight AC. The distance BC is measured from B, C being located along the line of sight AC. So for successive points D, E, etc., along the curve.*

5. The constant angle locus.

(1) Mark two points A and B with AB = 2 in. Place a set-square so that the sides containing the 60° angle fall on A and B. (See fig. 3.) Use successive positions of the set-square to plot the locus of the vertex of the 60° angle. Repeat for the 30° vertex. Cut an angle of 120° out of cardboard and repeat for the 120° vertex.

(2) State the property suggested by (1) in locus terms. What is (i) the direct theorem, (ii) the converse theorem in this locus? Which have you proved already?

The constant angle locus is used in navigation to keep ships clear of dangerous areas. For example, in fig. 21 the navigator is instructed to take bearings of the fixed objects A and B and to take care that the angle subtended by AB never exceeds a certain angle (say 130°); by doing this he will keep outside a certain arc which encloses the shoal.

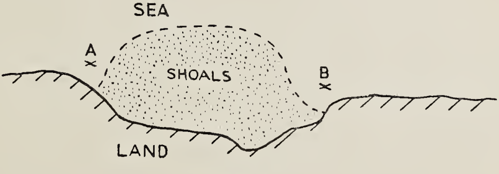

Fig. 13.21

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*For further detail consult a standard text on surveying, such as Davis, Foote and Rayner, *Surveying* (McGraw Hill, 1934).

When the base of a segment is given, how may a segment of a circle be constructed on this base to contain a given angle?

Suppose we begin with the segment as suggested in fig. 22, where the given angle x is acute. The angle at the centre will be $2x$ and hence the centre may be found by drawing an isosceles triangle with base angle v as shown in the figure. What is the value of y?

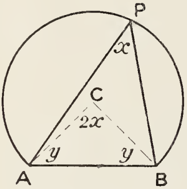

Fig. 13.22

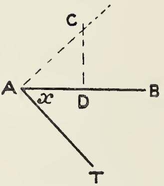

Fig. 13.23

Another method suggests itself. Draw a line at A which makes $\angle x$ with AB as in fig. 23. The line AT will be tangent to the required circle at A. Why? Hence the centre of the circle may be found by drawing the two perpendiculars shown dotted in the figure. Explain the construction.

EXERCISES. Constant Angle Locus

1. On a chord of 3.5 in. construct a segment of a circle to contain an angle of 70°. Measure the radius. Use both methods.

2. Repeat ex. 2 with chord of 7.24 cm. and angle of 110°.

3. A rhombus is made with four loosely jointed rods each 4 ft. long. One rod is fixed horizontally and the figure swings in a vertical plane. Construct the locus of the intersection of the diagonals.

4. A, B, C are three landmarks on a straight shore, AB = 10 miles, BC = 6 miles. From a ship at sea AB subtends an angle of 54° and BC an angle of 68°. Find the distance of the ship from B.

5. **ABC** is a triangle, **AB**=3.25 in., **BC**=4 in., **CA**=2.75 in. Find a point **O** within the triangle such that ∠**AOB**=106°, ∠**AOC**=120°. Measure ∠**BOC**.

6. Construct a triangle having:

	Base	*Altitude*	*Vertical angle*
(i)	4 in.	1 in.	90°
(ii)	10 cm.	2 cm.	120°
(iii)	8 cm.	5 cm.	90°
(iv)	3.5 in.	1 in.	54°

In each case measure the base angles.

Proof and Constructions

7. On the same base and the same side of it are drawn three triangles of vertical angle 30°. Show that a circle can be drawn to circumscribe all three triangles and that its radius is equal to the base.

8. Prove that the locus of the mid-points of chords of a circle which are drawn through a fixed point is a circle.

9. Of all triangles of given base and vertical angle, the isosceles triangle has greatest area.

10. **P** is a variable point on an arc **AB**. **AP** is produced to **Q** so that **PQ**=**PB**. Prove that the locus of **Q** is a circular arc.

11. Show how to construct a triangle given (i) base, vertical angle, and median, (ii) base, vertical angle, and area.

12. Show how to construct a quadrilateral **PQRS**, given **PQ**, **PR**, **PS**, ∠**QPS**, ∠**QRS**.

13. Show how to construct a quadrilateral **PQRS**, given **PQ**, **QR**, **RP**, **PS**, ∠**QSR**.

14. Show how to construct a parallelogram given the base and height, and the angle (subtended by the base) at the intersection of the diagonals.

6. The proof of the constant angle locus.

THEOREM. **Given an angle with (1) constant magnitude, (2) its sides through two fixed points R and S, one on each side, (3) its vertex on one side of RS. The locus of the vertex of the angle is a circular segment with base RS.***

The proof of the theorem falls into two parts of which II has already been indicated (§2).

I. *Data* $\angle$ RPS $= \angle x$.
   (1) $\angle x =$ constant.
   (2) R and S are fixed points.
   (3) P lies on one side of RS.

To prove that P lies on the segment of a circle with base RS.

II. *Data* A segment of a circle base RS.
   (1) P lies on the arc of the segment.

To prove that $\angle$ RPS $= \angle x =$ constant.

(It will be noted that (2), (3) of I are satisfied "automatically" if P lies on a segment with base RS.) The proof of I is an interesting example of the method of elimination (chap. I, §3).

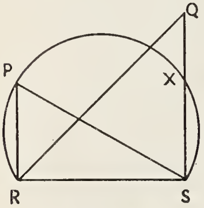

Fig. 13.24

Fig. 24 shows two positions, P and Q, of the vertex. The points R, P, S determine a circle, which according to the theorem should pass through Q.

Either this circle RPS passes through Q or it does not. In the latter instance there are various possibilities and in all the situation is as follows:†

The circle RPS must:

(i) cut SQ at Q,
(ii) cut SQ at X between S and Q,

---

*This theorem is a combination of Theorems 48 and 51. See the list at the end of the chapter.

†The variety of possibilities is, perhaps, surprising. It is reminiscent of the Sherlock Holmes passage in chap. IV.

(iii) cut SQ at X on SQ produced,
(iv) cut SQ at X on QS produced,
(v) touch SQ at S.*

The proof of I thus consists in eliminating each of the possibilities (ii)-(v). We illustrate with the elimination of (iv):—

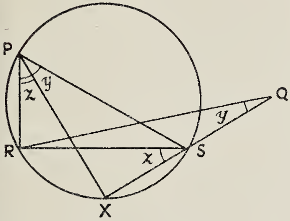

Fig. 13.26

P and Q are on the same side of RS as in fig. 26.

Since X is on QS produced, X lies on the opposite side of RS from P, i.e., X is on the arc of the segment alternate to the segment RPS. The line PX thus crosses the chord RS and hence ∠RPX < ∠RPS or $z<y$.

But ∠RQX $=y$ (data (1))
and ∠RSX $=z$. Why?

The exterior and opposite angle to ∠RQS with respect to ΔRQS is ∠RSX. Hence $z>y$, which contradicts the earlier finding. Hence the circle cannot cut QS produced.

You will wish to try eliminating (ii), (iii), (v) in a somewhat similar manner.

Are R and S on the locus? (See §1.)

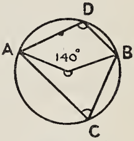

Fig. 13.27

7. The inscribed quadrilateral.

A quadrilateral which is inscribed in a circle has its four vertices on the circle. Such a quadrilateral is often called a *cyclic quadrilateral.*

(1) Find the magnitude of all the marked angles in fig. 27. What is the sum of the angles

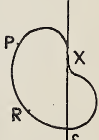

Fig. 13.25

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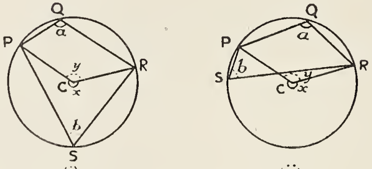
*Tangency is possible at S only since the circle is always to contain S. The assumption here is that a circle cannot meet a line in more than two points (distinct or coincident), i.e., such a figure as fig. 25 is not possible for a circle. In this figure the curve passes through RPS and is tangent to QS at X. The line QS cuts the curve once at S and at least three times at X. (Three coincident points.)

at the centre? of ∠s **ACB** and **ADB**? of ∠s **CAD** and **CBD**? What property of a quadrilateral inscribed in a circle does this suggest?



(i) (ii)

Fig. 13.28

(2) Fig. 28 shows an inscribed quadrilateral with the centre **C** (i) inside, (ii) outside, the quadrilateral.

How are a and x related? b and y? x and y? a and b?

What of the case when the centre **C** is (iii) on one side **RS** of the quadrilateral? Draw the figure.

State your findings as a theorem about an inscribed quadrilateral.*

(3) If **PQ** is produced to **T** in fig. 28 what is ∠**RQT**? The interior angle opposite to the exterior angle **RQT** is ∠**PSR**. State a theorem about these angles. Is the well-known theorem about any triangle thereby generalized to a theorem about any quadrilateral?

EXERCISES. The Inscribed Quadrilateral

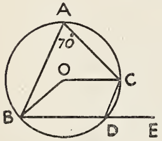


Fig. 13.29

1. In fig. 29, **O** is the centre of the circle and ∠**A** is 70°; find as many of the remaining angles in the figure as you can, giving reasons in each case.

2. **A**, **B**, **C** are three points on a circle such that ∠**ACB** = 122°. **O** is the centre. Prove that ∠**OAB** = 32°. At what angle will the tangents at **A** and **B** intersect? Prove your statement.

---

*This theorem was obtained earlier in another form. (Theorem 53.)

3. **ABCD** is a quadrilateral with the angles at **B** and **D** right angles. Draw the diagonals **AC** and **BD**. Given that the whole angle at **A** is 60° and $\angle$ **CBD** 40°, find the remaining angles in the figure.

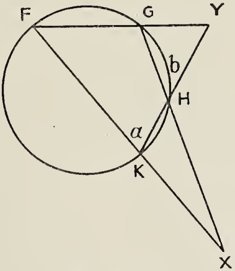


Fig. 13.30

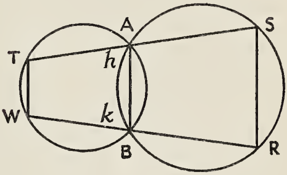


Fig. 13.31

4. Make freehand copies of the figures and write in each angle its size, when

(i) Fig. 30:—$\angle a=68^\circ$, $\angle b=36^\circ$.
(ii) Fig. 30:—$\angle a=90^\circ$, $\angle \mathrm{X}=41^\circ$.
(iii) Fig. 31:—$\angle h=85^\circ$, $\angle k=73^\circ$. Show that **RS** is || to **WT**.
(iv) Fig. 31:—$\angle \mathrm{S}=66^\circ$, $\angle \mathrm{R}=94^\circ$.

Proofs

5. A parallelogram inscribed in a circle must be a rectangle.

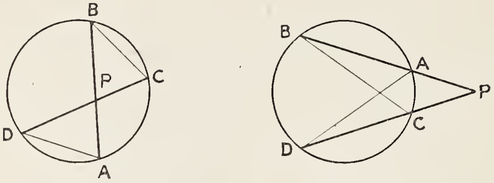
6. If a trapezium can be inscribed in a circle, it must be isosceles.

7. The sides **BA**, **CD** of a quadrilateral **ABCD**, inscribed in a circle are produced to meet at **O**; prove that Δs**OAD**, **OCB** are equiangular.

8. In fig. 31, express the other angles in terms of h and k, and hence prove **RS** parallel to **WT**.

9. In fig. 30, calculate angles a and b if $\angle \mathrm{X}=20^\circ$, $\angle \mathrm{Y}=34^\circ$. Also obtain a general relation connecting the angles **X**, **Y** and b.

10. Fig. 32 shows two chords **AB** and **CD** intersecting at a point **P** (i) inside, (ii) outside, the circle. Show that in either case Δ**BPC** and Δ**DPA** are equiangular. Hence, show that **PA.PB = PC.PD.***



(i) (ii)

Fig. 13.32

State this result as a theorem about any two intersecting chords of a circle.

Should the case (iii) **P** on the circle not be included in the theorem?

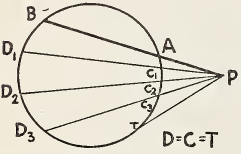


Fig. 13.33

11. Fig. 33 is a "movie" of Fig. 32(ii). **P** is a fixed point outside the circle and **BAP** is a fixed secant through **P.** The secant **DCP** pivots about **P.** Successive positions are shown. Apply the theorem of ex. 10 to show that $PT^2 = PA.PB$.

State this result in words.

8. Concyclic points.

A set of points through which a circle may be passed is said to be a *concyclic set of points*. Thus three points are usually concyclic. What is the exception? Again, the four vertices of a rectangle are concyclic, whereas the only rhombus whose vertices are concyclic is a square.

When are four points concyclic? More directly put, when may the quadrilateral of the four points be inscribed in a circle?

---

*Be careful to distinguish the product $x.y$ and the ratio $x:y$.

You have already encountered in this chapter *necessary* conditions for a cyclic quadrilateral, i.e., conditions which are accompaniments of a quadrilateral being inscribed in a circle. These are the following:

If the quadrilateral **PQRS** (fig. 34) is cyclic, then

(a) Opposite angles are supplementary (§7 (2))

$$\angle P + \angle R = \angle Q + \angle S = 180°.$$

(b) The angle formed by a side and a diagonal is equal to the angle formed by the opposite side and the other diagonal (§2).

e.g., $\angle SPR = \angle SQR$.

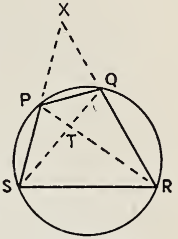


Fig. 13.34

(c)(d) Either of two conditions involving lengths is satisfied:

(i) Suppose that a pair of opposite sides intersect, viz., **SP** and **RQ** intersect in **X**.

(c) $SX.PX = RX.QX$. (§7, ex. 10)

(ii) Let the diagonals **PR** and **QS** intersect in **T**.

(d) $PT.RT = QT.ST$. (§7, ex. 10)

Are any of these conditions sufficient for a cyclic quadrilateral? In other words, will any one of these conditions guarantee that the quadrilateral **PQRS** is cyclic? It turns out that each of these conditions is a sufficient condition, so that the truth of any one of them guarantees the truth of each of the others.

The sufficiency of (b) has already been indicated in §6; **SR** serves as the base of the segment in §6 with **P** and **Q** as two positions of the vertex of the angle of constant magnitude.

The proofs of the sufficiency of (a), (c), (d) are proofs of converses and may be carried out in the spirit of §6.

EXERCISES. Concyclic Points

1. Draw freehand an acute-angled triangle ABC and its altitudes AD, BE, CF; assume that the altitudes all pass through a point H. Find all the sets of concyclic points and say whether each set depends directly on the fact that two angles are equal or on the fact that two angles are supplementary.

2. Repeat ex. 1 for an obtuse-angled triangle.

3. ABCD is a parallelogram. A circle drawn through A, B cuts AD, BC (produced if necessary) in E, F respectively. Prove that E, F, C, D are concyclic.

4. ABCD is a quadrilateral inscribed in a circle. DA, CB are produced to meet at E; AB, DC to meet at F. (Produce the sides to meet before putting in the letters.) Prove that, if a circle can be drawn through AEFC, then EF is a diameter of this circle; and BD is the diameter of the circle ABCD.

5. The straight lines bisecting the angles of any convex quadrilateral form a cyclic quadrilateral.

6. Show how to construct a cyclic quadrilateral ABCD, given AB, BC, CD, ∠B.

7. Explain how to construct a cyclic quadrilateral using only a protractor and a ruler.

XI. The Circle. Angle Properties

A. The Angle in a Semicircle

Terms: angle in a semicircle.

THEOREM 46. **The angle in a semicircle is a right angle.**

THEOREM 47. (CONVERSE OF THEOREM 46.) **Given a right angle with (1) its sides through two fixed points R, S, one on each side, (2) its vertex on one side of RS. The vertex of the angle lies on a semicircle with diameter RS.**

Theorems 46 and 47 may be combined into a single theorem stated in locus terms.

B. The Angle in a Segment

Terms: segment, alternate segments, major and minor segments, base and arc of segment, angle in a segment, angle at the centre, angle at the circumference.

Theorem 48. **The angle in a segment is half the angle at the centre subtended by the base of the segment.***

Theorem 49. **The angle in a segment is constant.**

Theorem 50. **The angle in a major segment is an acute angle, whereas the angle in a minor segment is an obtuse angle.**

Theorem 51. (Converse of Theorem 48.) **Given an angle with (1) constant magnitude, (2) its arms through two fixed points R and S, one on each arm, (3) its vertex on one side of RS. The vertex of the angle lies on a segment of a circle with base RS.**

Theorems 48 and 51 may be combined into a single theorem stated in locus terms. (See §6.)

Theorem 52. **(Tangency.) The angles which a tangent makes with a chord through its point of contact are equal to the angles in the alternate segments formed by the chord.**

Theorem 53. **The angles in alternate segments of a circle are supplementary.**

C. The Inscribed Quadrilateral. Concyclic Points

Terms: concyclic points; cyclic quadrilateral.

Theorem 54. **The opposite angles of an inscribed quadrilateral are supplementary.**

Corollary. **An exterior angle of an inscribed quadrilateral is equal to the interior opposite angle of the quadrilateral.**

---

*Caution: The angle at the centre must be properly defined. See figs. 9 and 10.

THEOREM 55. **The angle formed by a side and a diagonal of an inscribed quadrilateral is equal to the angle formed by the opposite side and the other diagonal.**

This theorem is merely a re-statement of Theorem 48.

THEOREM 56. **Let the diagonals of the inscribed quadrilateral PQRS intersect in T. Then the products of corresponding segments of the diagonals are equal, namely**

$$PT.RT = QT.ST.$$

This theorem may be re-stated in terms of two chords PR and QS which intersect inside the circle at T.

THEOREM 57. **Suppose that a pair of opposite sides of an inscribed quadrilateral intersect, viz., SP and RQ intersect in X. Then $SX.PX = RX.QX$.**

Theorem 57 may be re-stated in terms of two chords which intersect outside the circle.

The converse theorems for Theorems 54-57 are all true, as was noted in §8.

CHAPTER XIV

SPACE FIGURES

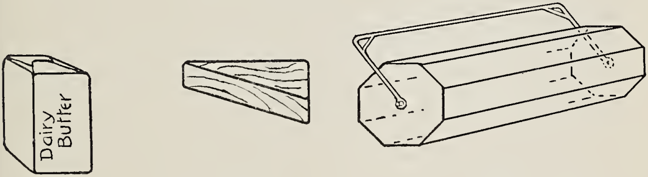
1. The primary solids.

Make a list with illustrations (magazine pictures) of ordinary space objects.

An examination of your collection will show that certain simple figures recur, singly or in combination, with surprising frequency:

(a) The prism.

The prism may have a square base (a lb. package of butter) a triangular base (a wedge-shaped door stop) or a many-sided base (an octagonal roller used in a certain laboratory to test mattress springs). These are illustrated in fig. 1.



(i) (ii) (iii)

Fig. 14.1

The simplest prism is the *cube*; the box figure (fig. 1(i)) is often called a *cuboid*. Prisms are either *right prisms* or *oblique prisms*. The right prism has its edges perpendicular to the bases; all the figures above are right prisms. Why? One of the simplest oblique prisms is the *parallelepiped*, with all its faces parallelograms. Crystals are often oblique prisms.

(b) The pyramid.

When the base of the pyramid is a regular polygon and the line joining the vertex of the pyramid to the centre of the base is perpendicular to the base, the pyramid is a (right) *regular pyramid*. A familiar example is the spire.

(c) The cylinder.

The commonest sort is the *right circular cylinder* (e.g., a food container). Can you draw a circular cylinder, not a right cylinder? Bases other than circles are easy to imagine.

The conduit shown in fig. 2 is a "cylinder." Its egg-shaped section has certain interesting flow properties.

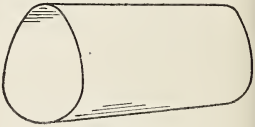


Fig. 14.2

(d) The cone.

The *right circular cone* is familiar. Can you draw a cone with a circular base, not a right cone? Draw a right elliptic cone.

(e) The sphere.

No comment is necessary.

Sometimes a space object is a *solid* like a steel axle, or again the object may be a container or *surface* like a gasoline drum. Both aspects (i) solid, (ii) surface, are important.

You will likely have noticed that some of the figures in your collection are composites of the simpler figures (a)-(e). Thus the ordinary garage is a composite of a cuboid and a right triangular prism; the famous stratosphere balloon *Explorer II*, when inflated at low altitudes looked like an ice-cream cone generously full. At high altitudes it became more nearly spherical.*

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*"Man's Farthest Aloft," *Nat. Geog. Mag.*, Jan., 1936.

2. Two theorems on equal volumes.

If we take a pile of equal rectangular cards and push the pile over sideways and endways, we form an oblique parallelepiped as in fig. 3. The opposite faces are parallel and each face is a parallelogram. Clearly the volume of the cards is unaltered, and so are the base and height of the pile. From this we obtain a theorem analogous to the theorem that parallelograms on the same base and between the same parallels are equal in area.

Parallelepipeds on equal bases and of the same height are equal in volume.

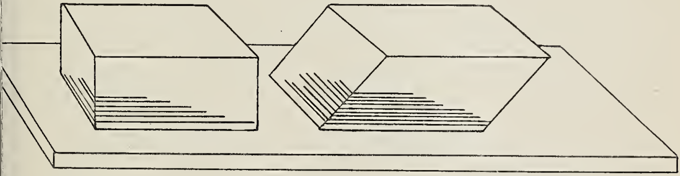


Fig. 14.3

Similarly, if we had a pile of cards forming a pyramid we could push them over and form another pyramid with the same base, height and volume.

Pyramids on equal bases and of the same height are equal in volume.

3. Prism and cylinder.

If the cross-sections of a solid by a series of parallel planes are all congruent, the solid is said to be *of uniform cross-section.* If the cross-section is a figure composed of straight lines, the solid is a *prism*; if the section is a closed curve, the solid is a *cylinder.*

A section of a prism or cylinder *normal* (or perpendicular) to the generators is called a *normal or right section.*

If the generators of the solid are perpendicular to the end sections, the solid is a *right* solid.

The area of the curved surface of a cylinder is found by cutting along a generator and developing the surface.

The volume of a cuboid (or rectangular block) is measured by the product of the length, breadth and height of the cuboid or by the product of the base-area and the height. These rules are particular cases of a rule that applies to any right solid of uniform cross-section (e.g., prisms, rails for railways, I-beams, cylinders, any metal goods made by rolling or pulling through a die).

The volume of a right solid of uniform cross-section is measured by the product of the area of the normal section and the height (or length).

By the first theorem of §2, we see that

The volume of an oblique prism or cylinder is the product of the base-area and the height.

The height of an oblique prism or cylinder is the perpendicular distance between the ends.

EXERCISES. Prism and Cylinder

1. What shapes are the normal sections of (i) a lead pencil, (ii) a typewriter case, (iii) an aeroplane strut?

2. Give other instances of prisms and cylinders and state what is the normal cross-section in each case.

3. Sketch the shape of the shadow cast on the ground by (i) a penny held horizontally, (ii) a sphere, the sun not being overhead.

√ **4.** Find the area of the whole surface and the volume of the prism whose plan and elevation are given in fig. 4.

√ **5.** A hollow right prism of height 3 ft. stands on a triangular base with equal sides sides of 1 ft.

(i) Find the area of its whole surface and its volume;

(ii) Find the length of the longest rod which can rest in the prism without projecting over the edge.

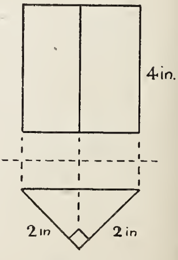


Fig. 14.4

✓ **6.** **AB**, **CD** are opposite generators of a circular cylinder of height 4 ft. and diameter 3 ft.

(i) Find the total surface area and volume of the cylinder;

(ii) Find the shortest distance from **A** to **D** along the surface of the cylinder.

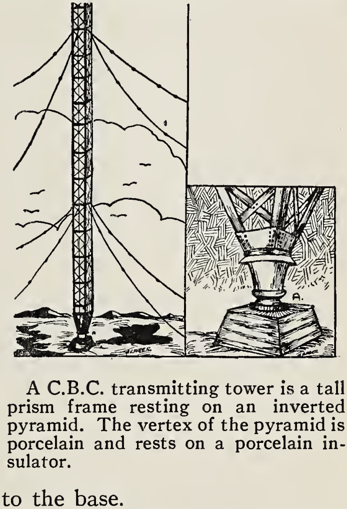
7. If a right circular cylinder has base-radius r and height h, show that its total surface area is $2\pi r(r+h)$ and its volume πr^2h.

4. Pyramids and tetrahedra.

You are already familiar with the fact that the section of a pyramid by a plane parallel to its base is similar to the base, and that the ratio of corresponding lines in the section and in the base is equal to the ratio of the distances from the vertex of the section and the base.

Note also that the pyramid cut off by the section is similar to the whole pyramid. The part of the pyramid between the base and a section parallel to the base is called a *frustum* the pyramid. (See fig. 5).

A pyramid is called *regular* if its *base* is a regular polygon, and the line joining its *vertex* to the centre of the base is perpendicular to the base.



A C.B.C. transmitting tower is a tall prism frame resting on an inverted pyramid. The vertex of the pyramid is porcelain and rests on a porcelain insulator.

If the base of a pyramid is a triangle, the figure is called a *tetrahedron*.

If the base of a tetrahedron is an equilateral triangle, and if the vertex is on the perpendicular to the base drawn through the centre of the equilateral triangle, the side faces of the tetrahedron are equal isosceles triangles. What we have called the centre of the equilateral triangle is its circumcentre, its incentre, the point of intersection of its medians and of its altitudes. (See fig. 6).

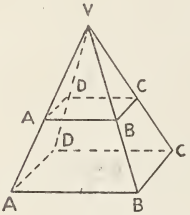


Fig. 14.5

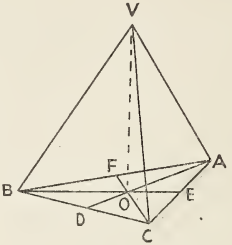


Fig. 14.6

EXERCISES. Pyramids and Tetrahedra

Trigonometry can be used with advantage

1. Make a sketch of a regular pyramid with vertex **V** and a square base **ABCD**; show **M** the centre of the base, and **N** the mid-point of **AB**. If the side of the base is 5 cm. and the height 6 cm.,

(i) draw accurately Δs **VMN**, **VAM**, **VAB**;

(ii) calculate or measure **VA** and the angles between **VA** and the base and between a side face and the base.

2. Show that the height of a regular tetrahedron of edge 3 in. is $\sqrt{6}$ in. (Remember that, if the medians **AD**, **BE** of Δ**ABC** meet at **O**, **DO** = 1/3 **DA**.)

3. **ABCD** is a regular tetrahedron, each edge of which measures 12 cm. **AE** = **AF** = 4 cm. (see fig. 7). The shaded area represents a section through **C**, **E**, **F**. Draw a side face of the tetrahedron; then draw the triangle **CEF** and find its area.

4. A tripod is composed of three rods, each of length 6 ft. It is placed so that the three points of support form an equilateral triangle of side 2 ft. Find the angle which each rod makes with the vertical, and the height of the apex of the tripod above the ground.

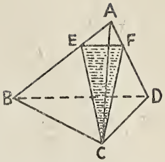


Fig. 14.7

✓ **5.** The eight edges of a pyramid which has a square base are each 2 in. in length. (i) Show that the height of the pyramid is $\sqrt{2}$ in.; (ii) Show that the perpendicular distance of each slant edge from the centre of the base is 1 in.

6. Three equal spheres of radii 5 in. are held in contact with each other on a horizontal plane and a fourth equal sphere is placed upon them. What is the height of the top of this fourth sphere above the plane, and the angle made with the vertical by the line joining the centres of the upper and a lower sphere?

5. The cone.

A cone bears to a pyramid the same relation as a cylinder bears to a prism, i.e., the sections parallel to the base are similar curves instead of similar-rectilinear figures.

In a *right circular cone* the base is a circle and the axis is perpendicular to the base. The generators in this case are all equal and their length is the *slant height* of the cone.

A section through the axis is an isosceles triangle and the angle at its vertex is called the *vertical angle* of the cone.

By slitting the curved surface of a right circular cone along a generator and developing it, a sector of a circle is formed, the radius of which is the slant height of the cone and the arc of which is the circumference of the base. (See fig. 8.) Hence **the area of the curved surface of a right circular cone is half the product of of the slant height and the circumference of the base.**

Fig. 14.8

If a cone is cut by a plane parallel to the base, one part is a cone similar to the whole cone and the other part is called a *frustum* of the cone.

EXERCISES. The Cone

1. Many familiar objects are conical frusta: a tree trunk, a parchment lamp shade, etc. Find other common examples.

2. (a) What is the section of a frustum of a cone by a plane containing the axis?

(b) Sketch a frustum of a cone and the figure obtained when the curved surface is slit along a generator and developed, showing which lengths correspond in the two figures.

3. A ring, diameter 24 in., is suspended by six equal strings from a point 5 in. above its centre, the strings being attached at equal intervals round its circumference. Find the length of each string and the angle between consecutive strings.

4. A paper semicircle, of which the bounding diameter is 6 in. in length, is folded so as to form a right circular cone. Find the height and vertical angle of the cone.

5. What area of canvas will be required for a conical tent, the diameter of the base being 20 ft. and the vertical height 15 ft.?

6. If the curved surface of a cone of radius r and slant l is developed into a sector, what fraction of 360° is the angle of the sector? Hence prove that the area of the curved surface is πrl.

---

7. The two ends of a frustum of a cone are of diameter 5 cm. and 12 cm., and its height is 9 cm. Find the slant height of the frustum and the slant height of the cone of which it is a part. Also find the length of the longest stick that could be placed in the frustum.

8. An electric lamp with a conical shade hangs over the centre of a table which measures 6 ft. by 8 ft. The semi-vertical angle of the cone forming the shade is 45°, the diameter of the rim of the shade is 2 ft. 6 in. and its vertex is 3 ft. above the centre of the table. Find the least distance below the vertex at which the electric bulb must be placed in order that the whole table may be illuminated.

9. A frustum of a right cone is of height h and the radii of its end faces are R and r. If the cone is rolled on a table, find the area of the circular ring on the table which has been rolled over.

6. Volume of pyramid and cone.

It has already been pointed out that, if two pyramids on equal bases have the same height, they are equal in volume.

Consider a pyramid whose base is a parallelogram. If we cut the pyramid in two by a section through the vertex and a diagonal of the base, we have two pyramids that are equal in volume.

Now consider fig. 9. The whole figure is a prism with a triangular base and its side faces are parallelograms. The figure is cut into three pyramids which are shown separately in fig. 10.

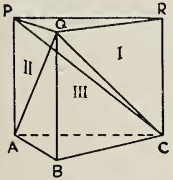


Fig. 14.9

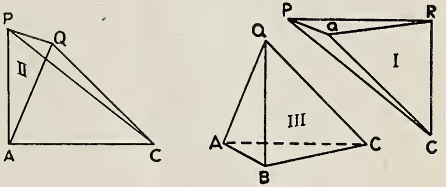


Fig. 14.10

If we take II and III together and regard C as vertex, their bases form a parallelogram with AQ as diagonal. Hence II and III are equal in volume.

Similarly I and II are equal in volume.

Hence the three pyramids are equal in volume and each is one-third of the prism.

Therefore, looking at I or III, we see that the following general result is proved for the particular case of a triangular pyramid.

The volume of a pyramid or cone is one-third of the product of the area of the base and the height.

The above proof can be extended to any pyramid, by dividing the base into triangles.

EXERCISES. Volume of Pyramid and Cone

1. Verify the truth of the formula for the volume of a pyramid by dividing a cube into pyramids by lines drawn from the centre of the cube to each of its corners.

2. A pyramid, 4 ft. vertical height, stands on a horizontal square base of 2 ft. side The top of the pyramid is cut off by a horizontal plane which bisects the axis. Find the volume of the whole pyramid and of the part cut off.

3. OX, OY, OZ are three straight lines at right angles to one another. A plane cuts them at A, B, C, so that $OA = 3$ in., $OB = OC = 4$ in. Find the volume of the tetrahedron OABC, the area of the triangle ABC, and hence the length of the perpendicular from O on ABC.

4. A right circular cone is cut from a solid sphere of radius a, the vertex and the circumference of the base being on the surface of the sphere. If x is the distance of the base from the centre of the sphere, prove that the volume of the cone is

$$\tfrac{1}{3}\pi(a^2-x^2)(a+x).$$

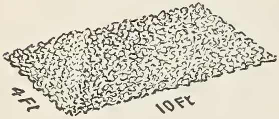


Fig. 14.11

5. Fig. 11 represents a pile of gravel. The base is a rectangle 10 ft. by 4 ft.; the ridge is 6 ft. long and 2 ft. above the base. Find the volume of the pile.

(Divide up the pile by vertical planes through the ends of the ridge parallel to the 4 ft. edge.)

6. A bobbin consists of two equal frusta of equal cones, the smaller ends of the frusta being coincident; the radius of each end of the bobbin is 5 in., that of the central section is 2 in.,

and the length of the bobbin is 12 in.; find (to within a cubic inch) the volume of the bobbin.

---

7. A tent is in the shape of a right pyramid of height 12 ft., its base being a regular hexagon whose sides are each of length 8 ft. Find its capacity in cubic feet (to the nearest cubic foot), and prove that the area of the canvas used in making the tent is twice the area of the base of the tent.

8. A heap of earth has a rectangular base 40 ft. by 60 ft., a level rectangular top 16 ft. by 36 ft., plane slant sides equally inclined to the horizontal, and a height of 20 ft. Find the volume of earth in the heap.

(Divide the solid, by vertical planes, into a cuboid, prisms and pyramids.)

7. The Sphere.

In chapter XI, §6, we have defined great circles and small circles.

In general, *two points*, A, B, *on a sphere determine a great circle uniquely*. For a plane is determined by A, B, and the centre of the sphere; and this plane cuts the surface in a great circle. The points A and B can be joined by two great-circle arcs, the minor arc and the major arc of the great circle.

The *shortest line* on the surface of a sphere connecting two given points is the minor arc of the great circle through the points, just as the shortest line in a plane connecting two points is the straight line. Accordingly, in the geometry of the spherical surface—*spherical geometry* as it is called—the great circle takes the place of the straight line in plane geometry.

The shortest-distance property of the great circle is used by navigators in great-circle sailing. Melbourne is approximately in the same latitude as Cape Town; but to sail along the parallel would be a much longer voyage than the great-circle route; this would run far to the south, as may be verified by stretching a piece of string on a globe between the two points. It is, in fact, impossible to follow the great-circle route throughout this voyage as it runs into the high latitudes, where navigation is unsafe.

If two points A, B on a sphere are at opposite ends of a diameter, it is no longer the case that only one great circle can be drawn through them. It is possible to draw an infinite number of planes containing the diameter, and each one of these cuts the surface in a great circle. We see therefore that through diametrically opposite points on a sphere an infinite number of great circles can be drawn (e.g., the meridians of the earth, drawn through the N. and S. poles).

Latitude and longitude are simply coordinates on the surface of a sphere (see chap. III, §3).

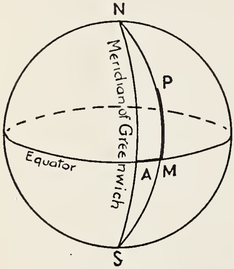


Fig. 14.12

The equator and the meridian of Greenwich, **NAS** (fig. 12), are taken as great-circle axes which cut one another at right angles; through the point **P** is drawn a great circle, at right angles to the equator, and therefore a meridian; the angles which **AM** and **PM** subtend at the centre of the earth are the longitude and latitude. Longitude is measured E. and W. of **A** through 180°; latitude is measured N. and S. through 90°. What is the latitude of the Equator? of the Poles?

EXERCISES. The Sphere

Take the earth's radius to be 4,000 *miles and* $\pi = 3.14$.

1. Draw a circle of radius 4 in. to represent the meridian of Greenwich; mark the position of the poles and the point at which it cuts the equator. Draw the radii of the parallels of latitude 30° N., $51\frac{1}{2}^\circ$ N., 70° N. Find the radii of these parallels.

2. (a) Find the length of an arc of one degree of longitude (i) at the equator, (ii) on the parallel $51\frac{1}{2}^\circ$. (Use the result of ex. 1.)

(b) A nautical mile is the length of an arc of one minute at the equator; use the result of (i) to express a nautical mile in land miles.

3. What is the distance between two places on meridian of longitude 5° W. whose latitudes are 30° N. and 55° N.?

4. Supposing it possible to travel along a great circle from a point 70° N. 90° E. to a point 70° N. 90° W., find the ratio of the length of the great-circle route to that of the route travelling due east. (Use the result of ex. 1.)

8. Surface and volume of sphere.

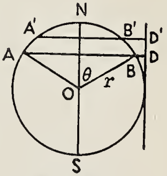


Fig. 14.13

Let the diameter NS (fig. 13) be at right angles to the parallel chords AB, A′B′; and let the figure rotate about NS.

The circle generates a sphere; A, B a small circle; the chords AB, A′B′ parallel circular areas; the arcs ANB, ASB *spherical caps*; the arcs AA′, BB′ a *spherical belt*.

It can be proved that

A spherical belt cut off by two parallel planes has the same surface as a belt cut by the same planes from a cylinder circumscribing the sphere and having its generators perpendicular to the planes. (See fig. 14.)

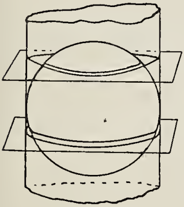


Fig. 14.14

Hence

The surface of a sphere $= 4\pi r^2$.

Let σ be the surface of any small region on the surface of the sphere; S the whole surface. A radius whose extremity travels round the boundary of this region will sweep out a cone whose vertex is at the centre. To the first order of small quantities we may regard this as a cone of height r standing on a plane base of area σ; its volume is $\frac{1}{3}r\sigma$.

Hence the volume of the whole sphere

$$\Sigma\tfrac{1}{3}r\sigma = \tfrac{1}{3}r\Sigma\sigma = \tfrac{1}{3}rS = \tfrac{4}{3}\pi r^3.$$

The volume of a sphere $= \frac{4}{3}\pi r^3$.

EXERCISES. Surface and Volume of Sphere

1. Prove that half the earth's surface is included between the parallels 30° N. and 30° S.

2. Find the latitude of the four parallels, two N. of the equator, two S. of it, which divide the earth's surface into five zones of equal area.

3. A boiler has the form of a right circular cylinder with two convex hemispherical ends. Show that the area of its external surface is equal to the product of its greatest length and the circumference of the circular section of the cylinder.

4. Find expressions for the curved surface of a segment of a sphere (i.e. a spherical cap) in terms of (i) R, the radius of the sphere, and h, the height of the segment; (ii) h and r, the radius of the base of the segment.

5. (a) A sphere rests on a table, and is covered by a hollow cone which also rests on the table and just touches the sphere. If the radius of the sphere be 3 in., and the height of the cone 8 in., find the volume of the cone.

(b) Find also the radius of the circle of contact between sphere and cone, and the ratio in which the surface of the sphere is divided by this circle.

6. Taking the earth as a sphere of radius 4,000 miles, find the distance from the equator of the parallel of latitude which is such that the section through it has half the area of the section through the equator.

9. The regular polyhedra.

Regular prisms and pyramids have already been defined. The *cube* is a special type of regular prism, since its faces are all congruent regular quadrilaterals (squares). Likewise the figure called the *regular tetrahedron* (four faces) is a special regular pyramid with its faces congruent regular triangles (equilateral triangles).

Such special regular figures are called *regular polyhedra*, defined as follows:

A regular polyhedron is a space figure (i) *in which the faces are congruent regular polygons and* (ii) *in which the same number of plane faces meet at each corner.*

It should be clear that unless a space figure with plane faces is re-entrant, **the sum of the plane angles that meet at any vertex must be less than four right angles.** Use this fact to answer the questions in (1).

(1) Is it possible to have a solid with three squares meeting at a corner? Is it possible with more than three squares?

Is it possible to have a solid with 3, 4, 5 or 6 equilateral triangles meeting at a corner?

Is it possible to have a solid with regular pentagons meeting at a corner? Repeat for regular hexagons.

Your answers in (1) should show you that **it is impossible to have more than five regular polyhedra.**

(2) Make freehand sketches of a regular tetrahedron and a cube.

Make a sketch of a regular octahedron (bounded by eight equilateral triangles) by joining the mid-points of the faces of a cube. (Draw the cube in pencil and the octahedron in ink.)

Fig. 15 shows three equal lines bisecting one another at right angles. Copy the figure freehand (in pencil) and draw a regular octahedron by joining the ends of the lines (in ink).

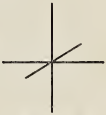
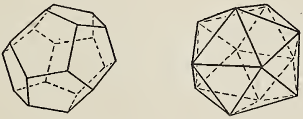


Fig. 14.15

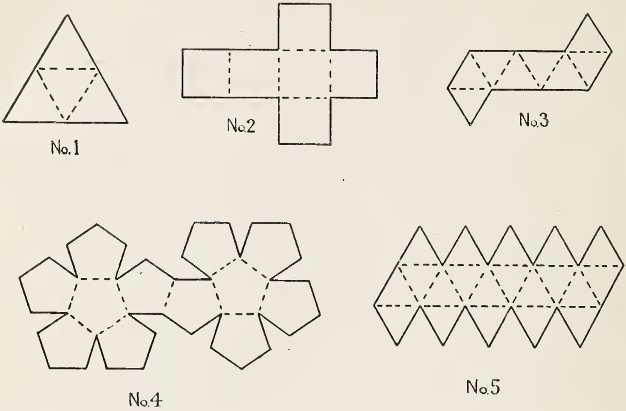
Fig. 16 shows the two other regular polygedra.



(i) Dodecahedron (i.e. 12-faced) (ii) Icosahedron (ie. 20-faced)

Fig. 14.16

You can make models of these figures from the patterns shown below (fig. 17).



No. 1 No. 2 No. 3 No. 4 No. 5

Fig. 14.17

The last book (Book XIII) of *The Elements*, Euclid's famous treatise on geometry, was concerned with these regular figures. Their study began with the Pythagoreans (c. 550 B.C.).*

It is interesting to note that many substances crystallize in the form of some of the regular solids, e.g.

Form	*Substance*
Tetrahedron	Blende, Tetrahedrite (grey copper ore).
Cube	Fluor-spar, Galena (lead sulphide).
Octahedron	Alum, Diamond,† Gold, Magnetite.

**Book XIII* is omitted from the *Home University Library* edition of Euclid, referred to previously.

†The natural faces are not to be confused with the artificial faces seen in cut stones. As a rule, the crystals of diamond and gold are waterworn and not very well defined in shape.

EXERCISES. The Regular Solids

1. Calculate the length of the diagonal of a regular octahedron in terms of the length of the edges.

2. Show that a regular octahedron can be divided into two square pyramids in three different ways.

3. If each corner of a cube were planed off symmetrically (see fig. 18), and the process continued till the faces of the cube disappeared, what solid figure would be constructed?

4. Show that the mid-points of the edges of a regular tetrahedron are the vertices of a regular octahedron.

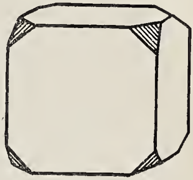


Fig. 14.18

10. Great Circle Geometry.

The great circle on the sphere, §7, is analogous to the straight line in the plane. The table which follows shows by comparison some of the properties of straight lines in the plane and of great circles on the sphere.

Plane	*Sphere*
(1) *Two points* P *and* Q *in the plane determine a unique straight line in the plane.*	(1) *Two points* P *and* Q *on the sphere determine a unique great circle on the sphere unless* P *and* Q *are the opposite ends of a diameter, when there are infinitely many great circles through* P *and* Q. (Ex. 1.)
(2) *Two straight lines in the plane either intersect in one point or they do not intersect. If they do not intersect they are said to be parallel to one another.*	(2) *Two great circles on the sphere always intersect in two points.* (Ex. 2.)

Plane	*Sphere*
(3) P *and* Q *are two points in the plane. The straight line segment* PQ *is the line of shortest distance in the plane between* P *and* Q.	(3) P *and* Q *are two points on the sphere. The minor arc of the great circle through* PQ *is the curve of shortest distance on the sphere between* P *and* Q. (Ex. 3.)
(4) *Three points* R, S, T *in the plane but not on a straight line determine a (plane) triangle* RST.	(4) *Three points* R, S *and* T *on the sphere but not on a great circle determine* 8 *(spherical) triangles* RST. *Each triangle has three sides (arcs of great circles) and three angles (angles between the tangents to each pair of great circles at* R, S *and* T). *Any one of the* 8 *triangles determines all the others.* (Ex. 4.)
(5) *The sum of the angles of any plane triangle is constant, being two right angles.*	(5) *The sum of the angles of any spherical triangle exceeds two right angles.* (Ex. 5.)

The comparison has been carried far enough to reveal a shocking state of affairs on the sphere! Straight lines (great circles) always intersect in two points, so that there are no parallel straight lines in spherical geometry. Moreover, the sum of the angles of a triangle always exceeds two right angles. Nor is this "mere speculation" for the spherical triangle is of practical importance in large scale surveys, in navigation and in astronomy.

You will no doubt wonder if it is possible to have a geometry in which the sum of the angles of a triangle is always less than two right angles. It is possible, but that is another story.*

EXERCISES. Great Circle Geometry

1. (a) Make a drawing to show a series of planes all passing through a single line. Model this in cardboard.

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*Young, *Monographs on Modern Mathematics (Non-Euclidean Geometry)*. (Longmans, Green, 1915.)

(b) How many of these planes pass through a single point not on the line?

(c) How many great circles pass through a single point **P** on the sphere? Illustrate this on the earth's surface.

(d) How many of the great circles through **P** also pass through **Q**? Always? Illustrate on the earth.

(e) Prove the property in (d); state the assumptions on which your proof rests.

2. (a) If two planes have a point in common, what else do you know?

(b) Assume (a). Use (a) to prove that two great circles on a sphere intersect in two points.

3. (a) Can you prove that the shortest distance between two points in the plane is a straight line? What are your assumptions?

(b) What is "great circle sailing"? On such a course the bearing of the vessel constantly changes; can you see why? NOTE.—The navigator usually sails as far as possible with constant bearing, then changes course, sailing for a time with bearing constant, etc.

4. (a) Cut three balsa wood strips 6 in. long. Gum small stickers on each strip near the end of the strip, with letters RR' for the first strip, SS' for the second and TT' for the third. Glue the strips together at their mid-points to form a double tripod.

(b) The six points $R, .. T'$ lie on a sphere. Why? Do the points RST determine a spherical triangle? Why $R'S'T'$?

(c) Note that you can stand the sticks upon RST or upon $R'S'T'$. Write the possible triads of points on which the sticks can stand. How many triads? How many spherical triangles? NOTE.—Remember that a spherical triangle must have *great-circle* arcs for its sides.

(d) You may wish to make a model of three equal wire hoops intersecting in the six points $R, .. T'$. The spherical triangles may be painted (coloured enamels) in pairs: RST, $R'S'T'$, etc.

5. (a) Draw a circle on cardboard, radius 3 in. Cut away a quarter of the circle, and fold the remainder to form fig. 19. The points **RST** lie on a sphere. Why? Describe the spherical triangle **RST**. What of its sides? Its angles? What is the sum of the angles?

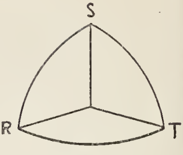

Fig. 14.19

(b) Use the model 4(d) to estimate the sum of the angles of each of the triangles. The angles can be transferred to a protractor by use of a bent tin strip, as indicated in fig. 20. **O** is the centre of the sphere, and the spherical angle is **SRT**. Remember that the angle at **R** is the angle between the tangents to the great circles **SR** and **TR** through **R**.

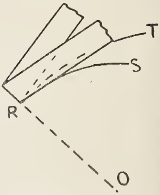

Fig. 14.20

11. Areas of similar figures.

Consider any figure, rectilinear or curvilinear; let it be divided up into small equal squares for purposes of counting the number of units of area. (See chap. VIII.) Now suppose that the figure, squares and all, is magnified, say 3 times; we shall have a figure similar to the former, the linear dimensions of the two being in the ratio 3:1. The area of each of the small squares will be magnified 9 times; and as the number of squares is unchanged, the area of the whole figure will be magnified 9 times. This suggests

The ratio of the areas of similar figures is equal to the square of the ratio of their linear dimensions.

This principle holds when the figures are space figures. Thus for the hull of a ship and its scale model the ratio of the areas of the hull surfaces (or any corresponding portions thereof) is equal to the square of the scale ratio. If the scale is 1:100, the area of the surface of the model hull is 1/10,000 that of the ship hull.

EXERCISES. Areas of Similar Figures

1. How is the area altered if the sides of (i) a square, (ii) a rectangle, (iii) a right triangle, (iv) any triangle, are (a) doubled, (b) halved, (c) trebled? Verify graphically with freehand sketches.

2. What is the ratio of the areas of two circles whose radii are (i) R, r, (ii) 3 in., 2 in.?

3. A plan of a hall is drawn on a scale of 10 ft. to 1 in. What is the ratio of corresponding areas on the plan and in the hall? What area is represented by 2.5 sq. in. on the plan?

4. What is the ratio of the areas of two similar triangles (i) on bases of 3 in. and 4 in., (ii) on bases a in. and b in.?

5. (a) The area of a triangle with a base of 12 cm. is 60 sq. cm.; find the area of a similar triangle with a base of 9 cm.

(b) What is the area of a similar triangle on a base of 9 in?

6. The areas of two similar triangles are 100 sq. cm. and 64 sq. cm.; the base of the greater is 7 cm.; find the base of the smaller.

7. What is the ratio of the area of a room to the area by which it is represented on a plan whose scale is 1 ft. to 1 in.?

8.* On a map whose scale is 1 mile to 1 in., a piece of land is represented by an area of 20 sq. in.; what is the area of the land?

9. On a map whose scale is 2 miles to 1 in., a piece of land is represented by an area of 24 sq. in.; what is the area of the land?

10. What is the acreage of a field which is represented by an area of 3 sq. in. on a map whose scale is 1 mile to 25 in.? (640 acres = 1 sq. mile.)

11. The height of the Great Pyramid is 149 metres; an exact model of the pyramid is made of height 1.49 metres, its side faces being triangles similar to the side faces of the pyramid. What is the ratio of the total slant surface of the pyramid to that of the model?

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*See local maps; measure areas in this fashion.

12. A model of a statue is made on a scale of 5 ft. to 2 in. The whole surface area of the model is found to be 14 sq. in. What is the surface area of the statute?

13. An exact model of a ship 450 ft. long is of length 10 in. The surface area below the water line of the model is 50 sq. in. What is the surface area exposed to water in the ship?

14. Two similar windows are glazed with small lozenge-shaped panes of glass, these panes being all identical in size and shape. The heights of the windows are 10 ft. and 15 ft. The number of panes in the smaller window is 1,200; what is the number in the larger?

15. The model of an astronomical observatory is made on the scale 1 : 50. The dome is a hemisphere of 40 ft. diameter. (i) What is the diameter of the dome on the model? (ii) How do the surfaces of the domes compare in area? (iii) How do the areas of the openings for the telescope compare in model and observatory?

12. Volumes of similar figures.

Consider any cuboid, its volume is a certain number of cubic units, which number depends on the length, breadth, and height of the cuboid. If we have a similar cuboid its length, breadth and height will each be altered in the same ratio. Hence if this ratio is $n:1$ the volume will be increased in the ratio of $n^3:1$, i.e., the new volume will be n^3 times the original volume. If the ratios of the lengths, breadths and heights are each $\frac{a}{b}$ then the volume will be $\frac{a^3}{b^3}$ times the original volume. This suggests the general principle

The ratio of the volumes of similar solids is equal to the cube of the ratio of their linear dimensions.

This principle is true regardless of the shape of the solids.

EXERCISES. Volume of Similar Figures

1. If the edges of (i) a cube, (ii) a cuboid, are (a) doubled, (b) trebled, (c) made n times as large, how is the volume altered?

2. What is the ratio of the volumes of two similar solids whose heights are 2 ft. and 3 ft.?

3. What is the ratio of the volumes of two similar solids whose heights are 4 ft. and 7 ft.?

4. Two similar solids have volumes of 125 cu. in. and 512 cu. in. A side of one is 3.5 in., what is the corresponding side of the other?

5. The shape of a box is to be kept the same but its capacity is to be doubled. By what must each side be multiplied to get the new dimensions?

6. Sand falling for 1 min. has formed a certain conical heap. After what additional time will the conical heap be twice as high?

7. A measure of diameter $3\frac{1}{2}$ in. holds 1 pint. What is the diameter of an exactly similar measure which holds 1 gallon?

8. A model of a dock is constructed on a scale of 1 in. to 50 ft. The model holds 1 pint of water; how many gallons of water could the dock hold?

9. Two exactly similar cylinders have volumes of 350 c.c. and 500 c.c. Find the ratio of their diameters in the form 1 to n.

10. In two class-rooms of the same shape the same fraction of the floor-space is occupied by chairs. Which room contains the larger volume of air per student?

11. An automobile tire company shows a picture of two tires rolling along a road to indicate growth in business during two successive five-year periods. If business over the second period was double that of the first, what should be the tire diameters in order to convey the correct impression?

12. A coal company shows growth in sales by a picture of two truckloads of coal, one truck being proportionately larger than the other. If sales have trebled, what should be the length-ratio for the trucks?

13. The structure of formulas.

The structure of the formulas of this chapter will have been noted. The formulas contain variables representing lengths, areas and volumes, together with certain constants (2, $4/3$, π, etc.).

It is useful to see how these formulas are balanced with respect to the variables. Thus the area A sq. in. of a circle is expressed in terms of the square of a length r in.

We say that an area is *two-dimensional with respect to length,* meaning that it requires for its expression the product of two lengths, a length being one-dimensional. When Jim writes $A=\pi r(r^2+h^2)$ for the lateral surface of a circular cone of radius r in. and height h in., his formula is wrong at a glance, because he has shown area to be three-dimensional with respect to length; the formula should have been $A=\pi r\sqrt{r^2+h^2}$, which is two-dimensional since $\sqrt{r^2+h^2}$ is one-dimensional, being the number of units of length in the slant height of the cone.

Volume is three-dimensional. Marian is wrong in stating the volume of the sphere as $4\pi r^2$. Why?

The formulas in the summary which follows should be checked for dimensional balance.

XII. Space Figures

A. PRISM AND CYLINDER

Terms: Right prism (e.g., cube, cuboid); oblique prism (e.g., parallelepiped).

Right cylinder (e.g., right circular cylinder); oblique cylinder.

Base, height, generator, normal section.

Notation: Height h; length of generator s; base radius r; base area b; perimeter of normal section p; lateral surface A; volume V.

Formulas: Prism or cylinder. $A=ps$, $V=bh$.
Right circular cylinder. $A=2\pi rh$, $V=\pi r^2h$.

B. PYRAMID AND CONE

Terms: Right pyramid; regular pyramid (e.g., spire); oblique pyramid (e.g., tetrahedron).

Right cone (e.g., right circular cone); oblique cone.

Vertex, base, height, slant height, generator, frustum.

Notation: Height h; slant height of regular pyramid or right circular cone s; base perimeter p; base area b; lateral surface A; volume V.

Formulas:

Pyramid or cone		$V=\frac{1}{3}bh.$
Regular pyramid	$A=\frac{1}{2}ps.$	
Right circular cone	$A=\pi rs,$ $s^2=r^2+h^2,$	$V=\frac{1}{3}\pi r^2h.$
Frustum, regular pyramid	$A=\frac{1}{2}(p_1+p_2)s$	$V=\frac{1}{3}(b_1+b_2+\sqrt{b_1b_2})h.$
Frustum, right circular cone	$A=\pi(r_1+r_2)s$	$V=\frac{1}{3}\pi(r_1^2+r_2^2+r_1r_2)h.$

C. SPHERE

Terms: Centre, radius, segment (cap), belt (zone).

Notation: Radius r; surface area A; volume V; height of zone h.

Formulas:

Sphere	$A=4\pi r^2.$	$V=\frac{4}{3}\pi r^3.$
Spherical belt	$A=2\pi rh.$	

D. REGULAR POLYHEDRA

The five regular polyhedra are (i) tetrahedron (4-faced), (ii) cube (6-faced), (iii) octahedron (8-faced), (iv) dodecahedron (12-faced), (v) icosahedron (20-faced).

E. SIMILAR SPACE FIGURES

When two similar space figures are given with corresponding lengths L_1 and L_2, corresponding areas A_1 and A_2 and corresponding volumes V_1 and V_2, then

$$\frac{A_1}{A_2}=\frac{L_1^2}{L_2^2}, \qquad \frac{V_1}{V_2}=\frac{L_1^3}{L_2^3}.$$

REVISION EXERCISES

A. Distance and Direction

These exercises should be solved by (a) *a scale drawing,* (b) *trigonometry. In* (b) *you may need to use unknowns for distances, such as h ft., x yd., etc.*

1. Find the height of a poplar tree from the following observations taken from two points **A** and **B** 100 ft. apart on opposite sides of the tree, and each 5 ft. above the level of the foot of the tree: angles of elevation of the top of the tree from **A** 35°, from **B** 56°.

2. The scale of a map is 1 mi. to 1 in. Two points on contours 100 ft. and 900 ft. are distant 1 in. on the map. What is the angle of slope of the line joining these points?

3. Construct a right-angled triangle **ABC**, given $\angle A = 90°$, $AB = 7$ cm., distance of **A** from $BC = 2.5$ cm. Measure the smallest angle.

4. Construct a triangle with height 2 in. and with angles at the extremities of the base of 40° and 60°. Find the length of the base.

5. Show that if two pairs of railway tracks of the same gauge cross, the figure formed by their intersection is a rhombus. If the angle of intersection is 30°, and the gauge is 4 ft. 8.5 in., find the side of the rhombus.

6. A straight, level railway line runs due east and west. **B** and **C** are points on the line 300 yd. apart. The bearings of a farmhouse **A** on the same level as the railway are N. 48° E. from **B** and N. 28° W. from **C**. Find the distances of the farmhouse from **B** and **C** and its distance from the railway line.

7. From a ship **S** the bearings of two other ships **P** and **Q** are respectively 35° and 313°. The distances of **P** and **Q** as observed

with a range-finder are 12,500 yd. and 15,700 yd. Find the distance and the bearing of **P** from **Q**.

8. In surveying a three-sided field **ABC**, a surveyor takes the following bearings: **B** from **A**, 64°; **C** from **B**, 205.5°; **C** from **A**, 152°. The side **AB** he finds by measurement to be 209 ft. Find the magnitudes of the angles and the other sides of the field.

9. The angle of elevation of a peak is 48°. From a second point 400 yd. nearer the peak up a slope inclined at 15° to the horizontal, the angle of elevation is 62°. Find the vertical height of the peak above the first point of observation.

10. From a point on the line joining the feet of two equal masts 210 yd. apart, on level ground, the angles of elevation of the tops of the masts are 30° and 42°. Find the height of the masts.

11. Is it possible to carry a stretcher measuring 10 ft. by 2 ft. round a right angle bend in a passage 5 ft. wide, if the stretcher must be kept horizontal?

12. The essential parts of a steam governor consist of two rods **AB**, **AD** each of length 5 in., freely jointed at **B**, **D** to two rods **CB**, **CD** each of length 7 in. **A** is fixed and **C** is free to move along the line **AC** and the whole rotates about **AC** as axis. The distance **BD** is 7 in. when at rest and at a high pressure of steam **B** and **D** have moved outwards till the distance **BD** is 9 in. Find through what distance **C** has moved along **AC**.

13. A man standing on a straight bank of a river sees, in a direction making an angle of 60° with the bank, a log floating down the river. After 2 min. the log is directly opposite to him. He now walks away from the river at right angles to the bank, a distance of 500 ft. in 2 min. and then the direction of the log makes an angle of 15° with the path along which he has walked. Find the speed of the current in feet per minute.

Three Dimensions

14. **A** is a point 100 yd. due W. of a tower, **B** is a point 50 yd. due S. of **A**; from **B** the angle of elevation of the top of the tower

is 20°. If A, B and the foot of the tower are on the same horizontal level, find the height of the tower.

15. From A a wireless mast is due W. and distant 300 ft. From B, due N. of A on level ground, the angle of elevation of the top of the mast is 32° and its bearing is 220°. Find the height of the mast.

16. An aeroplane is flying due E. at a height of 2000 ft. above the level of an airport. This is seen from the aeroplane to be due S. at an angle of depression of 29°, and 15 sec. later it is seen to be due S.W. Find the speed of the aeroplane in feet per second.

17. A ship sailing due E. at 6 sea miles an hour observes an unknown peak bearing 16° E. of N. at an elevation of 24°. Half an hour later, having kept on the same course, she observes the peak 24° W. of N. Find the approximate height of the peak. [1 sea mile = 6,080 ft.]

B. TRIANGLES; POLYGONS

Proofs

18. ABCD is a parallelogram in which $\angle A = 60°$. From AD cut off AE = AB; join BE and produce it to cut CD produced at F. Prove that ΔDEF is equilateral.

19. X is the vertex of an isosceles right-angled triangle XYZ; YR bisects $\angle$ Y and meets XZ at R; RN is drawn perpendicular to YZ. Prove that NZ = XR.

20. In a triangle ABC, the bisector of the angle A and the perpendicular bisector of BC intersect at a point D; from D, DX, DY are drawn perpendicular to the sides AB, AC produced if necessary. Prove that AX = AY and BX = CY. [Join BD, CD]. Does it follow that AB = AC? (See W. W. R. Ball, *Mathematicla Problems and Recreations*, p. 42.)

21. ABCD is an equilateral four-sided figure. A point P is such that PA = PC. Prove that P, B, D are in one and the same straight line.

22. PS, the bisector of the angle P of a triangle PQR, cuts QR at S; through S, ST and SU are drawn parallel to PR and PQ, thus forming a quadrilateral TPUS; prove that TPUS is a rhombus.

23. ABCDE is a regular pentagon. AC and BD intersect at O. Prove that the figure AODE is a rhombus.

24. ABC is a triangle right-angled at A, with squares ACKH, BCED drawn on AC, BC (as in the figure of Pythagoras' theorem); prove that triangles BCK, ECA are congruent.

25. L, M are the mid-points of the parallel sides AB, DC of a trapezium ABCD. Prove that LM bisects the trapezium.

26. Prove that the bisectors of two adjacent angles of a parallelogram are at right angles to one another.

27. ABCD is a parallelogram and BE is the bisector of the angle ABC. DE, drawn perpendicular to BE, is produced (backwards if necessary) to meet BC produced in P. Prove that the triangle PDC is isosceles.

28. The altitudes of a triangle ABC intersect at a point H; HA is produced to K so that $HK = BC$; KM is drawn parallel to HC to cut BH produced at M. Prove that $\Delta MKH \equiv \Delta ABC$.

29. Prove that any line drawn through the point of intersection of the diagonals of a parallelogram and terminated by its sides is divided into two equal parts.

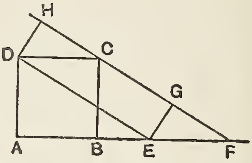
30. If RSTW is a quadrilateral such that RT bisects the angles at R and T, prove that SW is bisected by RT.

31. The bisectors of the base angles B, C of an equilateral triangle ABC meet at D. Through D lines are drawn parallel to the sides AB, AC of the triangle, meeting the base in E and F. Prove that E and F trisect BC.

32. ABCD is a square whose diagonals intersect at O. A circle is drawn with centre A and radius AO to cut AB, AD at P, Q. Prove that $PQ = AB$.

33. The medians BD, CE of △ABC intersect at G; prove that quadrilateral ADGE = △BGC. (Add to each a certain triangle.)

34. Show that if parallels are drawn to the diagonals through the vertices of a rhombus, they intersect to form a rectangle whose area is twice that of the rhombus.



35. In the given figure ABCD is a square and DEGH is a rectangle. Prove that they are equal in area.

36. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides is equal to the sum of the squares on the other pair of opposite sides.

Constructions

Example. If OX, OY, OZ *are three lines drawn from a point* O, *show how to construct a line through a given point* A *in* OX *which is terminated by* OZ *and bisected by* OY.

[In such a question first draw a freehand figure and show a line (ABC) which approximately satisfies the given conditions, i.e. imagine the problem done; then consider the properties of the figure and so seek a proper solution.

The fact that B is the mid-point of AC suggests the use of the mid-point theorem. If M is the mid-point of OA, MB is parallel to OC. Hence the following construction.]

Bisect OA at M; through M draw MB parallel to OZ to cut OY at B; join AB and produce it to cut OZ at C.

Then ABC is the required line.

An alternative construction can be made by using the theorem that the diagonals of a parallelogram bisect one another. In a freehand figure draw the parallelogram AOCD, where D is a point in OY.

37. Show how to draw a straight line equal and parallel to a given straight line and having its ends on two given straight lines.

38. In the equal sides AB, AC of an isosceles triangle ABC, show how to find points X, Y, such that $BX = XY = YC$.

39. If XOY is an acute angle, show how to find points A, B in OX, OY such that $\angle OAB = 3\angle OBA$. [What angle is equal to the sum of these angles?]

40. Through a given point O draw three straight lines OA, OB, OC of given lengths so that A, B, C may be collinear and $AB = BC$.

41. Draw a triangle ABC, produce CB to Q so that $BQ = BA$; produce BC to R so that $CR = CA$. Express the angles of ΔAQR in terms of A, B, C, the angles of ΔABC. Hence show how to construct a triangle when given its perimeter and its angles.

42. A and B are two fixed points in two unlimited parallel straight lines; show how to find points P and Q in these lines such that APBQ is a rhombus.

43. (i) A, B are two points on the same side of a straight line CD; in CD show how to find P such that $\angle APC = \angle BPD$;

(ii) If A, B are on opposite sides of CD, find P such that $\angle APC = \angle BPC$.

C. SIMILAR FIGURES

Calculations

44. The parallel sides of a trapezium are of lengths 4 in. and 12 in.; the other two sides are each divided in the ratio of 3 : 5 by a line parallel to the parallel sides. Calculate the two possible values of the length of this line.

45. A triangular framework ABC has $AB = 3$ ft., $BC = 3$ ft. 6 in., $AC = 2$ ft. 9 in. A rigid stick 2 ft. long is wedged between AB and AC so as to be parallel to BC. How far are its ends from A?

46. The diameter of the base of a cone is 20 in. and its height 15 in.; find the diameter of the section parallel to the base and 6 in. from it.

47. A rod **DC**, 2 ft. long, is suspended from two points **A**, **B** in the ceiling 4 ft. apart by two equal wires **AD**, **BC**, each 6 ft. long. The rod **DC** is now twisted round through 180° so that the wires cross at **O**. Calculate the length of **AO** and **OD**.

The rod is now untwisted into its original position and the points which were at **O** are joined by a string. What is the length of this string?

48. **ABC** is an isosceles triangle with base 4 in. and sides 6 in. A line is drawn from the middle point of the base perpendicular to one side. Find the distance of each vertex from this line.

49. A rectangular board 16 in. by 12 in. is sawn in two along a line perpendicular to a diagonal at a point 8 in. from one end of the diagonal. Calculate the length of the saw-cut.

50. **ABC** is an isosceles triangle with $AB = AC = 18$ cm., $BC = 10$ cm. A circle radius 2 cm. touches **AB**, **AC**. Find, by calculation, the distance of its centre from **A**.

51. **ABC** is a triangle inscribed in a circle; the tangent at **C** meets **BA** produced in **T**. If $TA = 5$ in., $AB = 3$ in., $AC = 4$ in., calculate **BC**.

52. **BAC** is a semicircle on **BC** as diameter and **AD** is perpendicular to **BC**. If $AD = 3$ cm., $BD = 5$ cm., find **CD** and the radius of the semicircle.

53. A right-angled triangle has sides of 18 cm. and 13.5 cm. At a point on the hypotenuse 9 cm. from the end of the longer side a perpendicular is drawn to the hypotenuse. Calculate the length intercepted on this perpendicular by the two sides.

54. Two beams **AOB**, **COD** each 10 ft. long cross at **O**, 6 ft. from the ends **A**, **C**, and rest with **A**, **C** fixed to the ground 5 ft. apart. A horizontal bar **PQ** 3 ft. long rests in the upper angle between the beams. What is the distance of **P** and **Q** from **O**?

55. The flywheel of an engine is 3 ft. 6 in. diameter and weighs 1800 lb. What is the weight of the flywheel in a smaller pattern of the same engine in which the flywheel has diameter of 2 ft.?

56. A rectangular plank measures 10 ft. by 9 in. What are the dimensions of a similar plank of half the area?

57. Draw any quadrilateral, and using only two set-squares, construct a similar quadrilateral.

58. Divide a circle into three parts of equal area by two concentric circles.

Proofs

59. AB, CD are the parallel sides of a trapezium whose diagonals intersect at O. Prove that Δs OAB, OCD are similar, and write down the three equal ratios of corresponding sides.

60. ABCD is a parallelogram, and any straight line through C cuts AB and AD produced at X and Y respectively; prove that AD : BX = DY : AB.

How must the line XCY be drawn to make the ΔsXBC and YDC congruent?

61. Show how to divide a triangle into four triangles each similar to the original triangle.

62. ABCD is any quadrilateral whose diagonals intersect at O. From O lines are drawn parallel to DC, DA meeting BC, BA in P, Q respectively. Prove that PQ is parallel to AC.

63. The bisector of the angle A of a triangle ABC meets the base in D. A parallel to DA through C meets BA produced in E. Prove that AE = AC and hence that DB : DC = AB : AC.

64. The bisector of the exterior angle at A of a triangle ABC meets the base BC produced in F. Prove that FB : FC = AB : AC. [Draw a parallel to FA through C.]

65. A triangle PQR is inscribed in a circle and the tangent to the circle at the other end of the diameter through P cuts the sides PQ, PR produced at H, K respectively; prove that the Δs PKH, PQR are similar.

66. AB, AC are equal straight lines. D and E are points, D in AB, E in AC produced, such that DB = CE; prove that DE is bisected by BC.

67. The straight lines joining the mid-points of opposite sides of a quadrilateral bisect one another.

68. **ABC** is an equilateral triangle. From **D**, the mid-point of **BC**, **DE** is drawn perpendicular to **AB**. Prove that the triangle **DEB** is one-eighth the area of the triangle **ABC**.

69. Through the vertices **A**, **B**, **C** of an equilateral triangle straight lines are drawn perpendicular to the sides **AB**, **BC**, **CA** respectively, so as to form another equilateral triangle. Compare the areas of the two triangles.

D. THE CIRCLE

Calculations

70. Two circles with centres **A** and **B** intersect in **C**. If $\angle ACB$ is 110°, find the angle between the tangents at **C** to the two circles. Give reasons.

71. **ABC** is a triangle inscribed in a circle. The tangent at **A** meets **BC** produced in **P**. If $\angle B = 46°$, $\angle C = 72°$, find $\angle P$, stating your reasons clearly.

72. **ABC** is a tangent to a circle at **B**, **BD** is a diameter and **BE**, **BF** are chords such that $\angle ABE = 15°$, $\angle CBF = 55°$; **DE**, **DF**, **EF** are joined. Find all the angles of the figure.

73. Draw a quadrilateral **ABCD** having $AB = 6.25$ cm., $BC = 8.4$ cm., $CD = 5.3$ cm., $\angle ABC = 95°$, $\angle BCD = 105°$. Measure **AD** and $\angle ADC$. Without further drawing, state whether **D** will be inside or outside the circle **ABC**. Give your reasons. Then test your conclusions by drawing the circle **ABC**.

74. A small wheel with 10 cogs drives a larger wheel with 35 cogs. When the small wheel has made one revolution, through what angle has the larger turned?

75. Six tiles in the shape of equilateral triangles of side 5 in. are placed flat around a point with one vertex of each at the point. A thin circular ring is then placed around the tiles. Find the area of the space between the ring and the tiles.

76. **OA** is the vertical line which is the junction between two walls of a room, **OB** and **OC** the horizontal lines running along the junction between the walls and the floor. What is the locus of the point **P** which moves about in the room under the following conditions:—(i) so as to be always 5 ft. above the floor? (ii) so that the angle **AOP** is always 50°? (iii) so as to be always equidistant from **OB** and **OC**? (iv) so as to be 4 ft. from **O** and 2 ft. from the plane **AOC**?

77. A bridge floor 80 ft. long is supported on a circular arch, the piers being 16 ft. below the floor level. Let r ft. be the radius of the arch, and b ft. be the height of the floor above the highest point of the arch. Find the relation connecting r and b. Calculate (i) b when $r=65$, (ii) r when $b=1$.

78. A chain passes around two wheels. The centres of the wheels are 20 in. apart. Find the length of the chain if

(a) the wheels are each of diameter 12 in.;
(b) the chain for (a) is crossed;
(c) the radii of the wheels are 4 in. and 10 in.

79. Three equal cylindrical pipes of 14 in. diameter are bound together, each touching the other two. What length of binding wire is necessary?

80. A weight on the end of a string 3 ft. long swings on each side of the vertical so that the greatest distance between its extreme positions is 2 ft. Through what angle does the string swing and what is the least vertical distance of the weight below the point of suspension?

81. A weight hangs by a string from the end **B** of a beam **AB**, 20 ft. long, and just rests on the ground when **AB** is inclined to the vertical at an angle of 42° (the end **A** being on the ground). If the beam is turned about **A** until **AB** is vertical, through what vertical distance is the weight raised?

82. Calculate the distance and angle of depression of the horizon seen from an aeroplane 15,000 ft. above the earth's surface. [Answer to nearest mile, take the earth's radius as 4,000 miles.]

83. The radii of the top and bottom of a can are 6 in. and 4 in. and the can is 14 in. deep. What is the radius of the largest ball which will just touch the bottom of the can?

Proofs

84. Chords AP, BQ are drawn perpendicular to a chord AB at its extremities. Prove that AP = BQ.

85. ABC is a triangle inscribed in a circle and BT is drawn parallel to AC cutting the tangent at A in T. Prove that $\angle ABC = \angle BTA$.

86. A chord BA of a circle is produced to meet the tangent at T in P. The tangent at A meets PT in Q. If $\angle APT = x$, $\angle BAQ = y$, prove that $\angle PTA = \frac{1}{2}(y - x)$.

87. QS, RT are altitudes of an acute-angled triangle PQR; prove that $\angle STR = \angle RQS$.

88. The triangle ABC is right-angled at A. If P is the centre of the square described on BC, prove that AP bisects $\angle BAC$.

89. A mechanism consists of a straight rod ABC, 8 in. long, and a rod BD, 4 in. long, hinged to ABC at B, the mid-point of AC. Prove that, however the mechanism is placed, AD is always at right angles to DC.

If D is fixed and A moves along a straight line DX, what is the greatest and least distances of A from D if DC cannot exceed 3 in.?

90. ABCD is a parallelogram. If the circle through the points A, B, C cuts CD in P, prove that AP = AD.

91. A circle whose centre is O is touched internally at A by a circle of half its radius. A radius OQ of the former circle cuts the smaller circle at P. Prove that arc AQ = arc AP.

92. State, without actually performing any construction, how you would solve the problem of drawing two tangents to a circle, which should include a given angle and intersect upon a given straight line. How many solutions of the problem would you expect to get?

93. **QR** is a chord of a circle, **TR** is the tangent at **R**; a straight line through **Q** perpendicular to this tangent meets it in **T** and the circumference of the circle again in **P**; **PM** is the perpendicular from **P** on **QR**. Prove that the angles **QPM**, **TPR**, **TMR**, **TRM** are all equal.

94. **ABCD** is a quadrilateral in a circle. One side **BC** is produced to **E**. Prove that the bisectors of the angles **BAD**, **DCE** meet on the circumference.

95. To two circles, centres **O** and **O′**, an internal and an external common tangent are drawn, meeting in **P**. Prove that **P** lies on the circle on **OO′** as diameter.

96. **O**, **A**, **B**, **C**, **D** are points on the circumference of a circle such that the angles **AOB**, **BOC**, **COD** are equal. Prove that the angle between the chords **AC**, **OB** is equal to that between **AD**, **OC**.

97. If a small circle is drawn to touch a large circle internally and to pass through the centre of the large circle, prove that any chord of the large circle which passes through the point of contact is bisected by the small circle.

98. Given a ruler for measuring lengths only, how can you determine whether four points are concyclic?

E. LOCI

Proofs

The double aspect of the proof of a locus can be compared to the proof of a theorem and its converse, since in each case part of what is given and part of what is to be proved are interchanged.

This double aspect of the proof should be clearly expressed in each exercise. The complete proof should be made in one or two instances. Otherwise begin with the conditions to be satisfied by **P** *and develop the first half of the proof. Sketch figures to show the extent of the locus. Describe the position of each locus clearly, e.g., if a circle, state its centre and radius.*

99. If **O** is a fixed point and **P** moves along a straight line, prove that the locus of the mid-point of **OP** is the right bisector of the perpendicular from **O** on to the given line.

100. Prove that the locus of the vertices of equivalent triangles on the same base and on the same side of it is a straight line parallel to the base.

101. A straight line AB of constant length moves with its extremities on two lines OL, OM at right angles to one another. If C be the remaining angular point of the rectangle whose sides are OA, OB, find the locus of C.

102. Prove that the locus of the mid-points of chords of a circle which are drawn through a fixed point is a circle.

103. P is a point on an arc of a circle, whose chord is AB. AP is produced to Q, so that PQ = PB. Prove that (i) the angle AQB is of the same size for all positions of P, (ii) the locus of Q is an arc of a circle.

The perpendicular bisector of AB meets the original arc in O; prove that O is the centre of the circle on which Q lies.

104. Given in position the right angle of a right-angled triangle and the length of the hypotenuse. Find the locus of the mid-point of the hypotenuse.

105. The circumcircle and base BC of an acute-angled triangle ABC are fixed. Perpendiculars are drawn from B and C to the opposite sides and cut each other at P. Prove that, as A moves on the circumcircle, the locus of P is an arc of a circle. Sketch the locus if $\angle A$ is obtuse.

106. AB is a fixed chord of a circle and RS is a variable diameter of the circle. AS, BR intersect at P. Find the locus of P.

107. A semicircular disc moves with the extremities of its diameter on two perpendicular straight lines. Show that the points on the semicircular arc describe straight lines.

Drawing

108. Plot the locus of points the ratio of whose distances from two fixed points, A and B is constant. (i) Take AB = 2 in., and the ratio 5 : 3; (ii) take AB = 3 in., and the ratio 2 : 1.

109. S is a fixed point 2 in. from a fixed line. P is a moving point, and X, M are the feet of the perpendiculars from S, P to the fixed line. Plot the locus of P when $SP = e.PM$ for the three values of e (i) 1, (ii) $\frac{1}{2}$, (iii) 2.

Use squared paper. Draw the fixed line down the middle of the paper, take X on this line near the middle of the paper. Label the lines parallel to the given line at $\frac{1}{2}$ in. intervals towards S, 0.5, 1, 1.5, ... With a pair of compasses mark the points where the locus cuts these lines: e.g. for (i) to find the point on the line labelled 1.5, put the compass point on S, and with the compass set at 1.5 in., by means of the squared paper, mark the two points where the circle cuts the 1.5 line. Mark also the point where the locus cuts SX.

In (iii) notice that there are points on each side of the fixed line.

The curves you will obtain are (i) a *parabola*, (ii) an *ellipse*, (iii) a *hyperbola*. S is a *focus*, and the fixed line a *directrix*.

110. In the loci in ex. 109 there is symmetry about SX. Is there also symmetry about any other line for any of the curves? If so, is there another possible position of a focus and directrix?

111. A and B are two fixed points 2 in. apart. Trace the locus of points P such that $PA + PB = k$, where (i) $k = 2.6$ in., (ii) $k = 4$ in.

[Two drawing pins and a piece of thread will provide a mechanical means of plotting the locus.]

112. Repeat ex. 111 with $PA - PB = k$, where (i) $k = 1$ in., (ii) $k = 1.5$ in.

F. PROOF

113. In order to inscribe a circle in a triangle RST, the angles at S and T were bisected, their bisectors intersecting at O. OU was drawn perpendicular to TS. A circle was then drawn with centre O and radius OU. Finally, OV and OW were drawn perpendicular to RT and RS.

(a) Which of the following conclusions do you believe to be now justified?

- (A) We have proved that the circle **UVW** is inscribed in Δ**RST**.
- (B) The circle will not touch the triangle at **V** and **W**.
- (C) Thus far we have not proved that the circle touches the triangle at **U**, **V** and **W**.

(b) Select the reasons from the list below which support the conclusion of your choice.

(1) Everybody knows that the point of intersection of the bisectors of two angles of a triangle is the centre of the inscribed circle.

(2) **OV** and **OW** were not constructed as radii.

(3) When we circumscribe a circle about a given triangle the side of the triangle is bisected and hence **TU** = **SU**.

(4) It is obvious that with radius **OU** the circle will touch the sides **RS** and **RT**.

(5) So far we have not proved that **OV** and **OW** are radii.

(6) The angles **S** and **T** may not have been bisected accurately.

(7) Anyone can see that the circle **UVW** touches Δ**RST** at three points.

(8) It would be purely accidental with the given construction if the circle touched **RT** at **V** and **RS** at **W**.

114. John was asked to prove that the right bisector of any chord of a circle passes through the centre of the circle. His proof (figure omitted) is given below. His figure showed **O** on **CD**.

(a) criticize John's proof; (b) what is circular reasoning?

Problem. Prove that the perpendicular bisector of a chord of a circle passes through the centre of the circle.

Data. a circle, centre **O**.
A chord **AB**, mid-point **C**.
The perpendicular bisector of the chord is **CD**.

To prove that **O** lies on **CD**.

Construction. Join **OA**, **OB**.

Proof. Δs **ACO, ABO.**

$$AC = CB, \quad \text{(Data)}$$
$$OC \text{ is common,}$$
$$\angle OCA = \angle OCB.$$
$$\therefore \Delta ACO \equiv \Delta ABO.$$
$$\therefore \quad OA = OB.$$

But since **OA = OB, O** lies on the right bisector of **AB.** (Theorem 26.)

The centre of the circle thus lies on the perpendicular bisector of any chord.

115. A hockey game has just been played between two school teams from *Brenner* and *Callum.* Three of the boys express their opinions of the game as follows:

A.—The score was *Brenner* 2, *Callum* 3, but one of *Callum's* goals was scored just as the whistle blew for a penalty. The referee quite unfairly allowed the goal. One of our best defence men was absent from the game. The referee gave 3 penalties to our players and only 1 to *Callum.* That was very unfair. The other team played much rougher hockey than did our boys. The referee was playing against us! Given a fair chance, *Brenner* would have won the game and would have shown themselves to be the better team.

B.—*Callum* is the better team. We play better combinations and on the whole are faster. The score, 3-2 in our favour, tells the story. Our goalie made more saves. The game was played cleanly by our boys; we only had one penalty and *Brenner* had three. The *Brenner* boys are poor sports. See how they protested the referee's decision over that goal! They always grumble when they do not win. The referee was a little slow but he didn't favour our side.

C.—The game was closely played from the word "go!" The *Callum* team has a good goalie. We lacked one of our best players. The referee was slow in stopping the play and that made both sides angry with him. However, the play changed sides quite often; neither team seemed to have the edge. The score shows that the teams were quite evenly matched.

(a) Which of A, B or C made or implied the statements which follow? Which of A, B or C made or implied contradictory statements to these? Which of these statements are statements of conclusion?

(1) The number of goals scored is a fair indication of the relative merits of the teams.
(2) Judging by the score neither team had the edge.
(3) The decisions of a referee may favour one team.
(4) The number of penalties awarded a team indicates the roughness of their play.
(5) A protest against the decision of a referee indicates poor sportsmanship.

(b) Analyse the opinions expressed by A, B and C to show (i) the data, (ii) the conclusions or findings, (iii) the steps, if any, in the reasoning from (i) to (ii), (iv) the assumptions upon which the reasoning is based, (v) any irrelevant statements.

SQUARES AND SQUARE ROOTS

n	n^2	$\sqrt{n}$	$\sqrt{10n}$	n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1	3·1623	**51**	26 01	7·1414	22·593
2	4	1·4142	4·4721	**52**	27 04	7·2111	22·804
3	9	1·7321	5·4772	**53**	28 09	7·2801	23·022
4	16	2	6·3246	**54**	29 16	7·3485	23·238
5	25	2·2361	7·0711	**55**	30 25	7·4162	23·452
6	36	2·4495	7·7460	**56**	31 36	7·4833	23·664
7	49	2·6458	8·3666	**57**	32 49	7·5498	23·875
8	64	2·8284	8·9443	**58**	33 64	7·6158	24·083
9	81	3	9·4868	**59**	34 81	7·6811	24·290
10	1 00	3·1623	10	**60**	36 00	7·7460	24·495
11	1 21	3·3166	10·488	**61**	37 21	7·8102	24·698
12	1 44	3·4641	10·954	**62**	38 44	7·8740	24·900
13	1 69	3·6056	11·402	**63**	39 69	7·9373	25·100
14	1 96	3·7417	11·832	**64**	40 96	8	25·298
15	2 25	3·8730	12·247	**65**	42 25	8·0623	25·495
16	2 56	4	12·649	**66**	43 56	8·1240	25·690
17	2 89	4·1231	13·038	**67**	44 89	8·1854	25·884
18	3 24	4·2426	13·416	**68**	46 24	8·2462	26·077
19	3 61	4·3589	13·784	**69**	47 61	8·3066	26·268
20	4 00	4·4721	14·142	**70**	49 00	8·3666	26·458
21	4 41	4·5826	14·491	**71**	50 41	8·4261	26·646
22	4 84	4·6904	14·832	**72**	51 84	8·4853	26·833
23	5 29	4·7958	15·166	**73**	53 29	8·5440	27·019
24	5 76	4·8990	15·492	**74**	54 76	8·6023	27·203
25	6 25	5	15·811	**75**	56 25	8·6603	27·386
26	6 76	5·0990	16·125	**76**	57 76	8·7178	27·568
27	7 29	5·1962	16·432	**77**	59 29	8·7750	27·749
28	7 84	5·2915	16·733	**78**	60 84	8·8318	27·928
29	8 41	5·3852	17·029	**79**	62 41	8·8882	28·107
30	9 00	5·4772	17·321	**80**	64 00	8·9443	28·284
31	9 61	5·5678	17·607	**81**	65 61	9	28·460
32	10 24	5·6569	17·889	**82**	67 24	9·0554	28·636
33	10 89	5·7446	18·166	**83**	68 89	9·1104	28·810
34	11 56	5·8310	18·439	**84**	70 56	9·1652	28·983
35	12 25	5·9161	18·708	**85**	72 25	9·2195	29·155
36	12 96	6	18·974	**86**	73 96	9·2736	29·326
37	13 69	6·0828	19·235	**87**	75 69	9·3274	29·496
38	14 44	6·1644	19·494	**88**	77 44	9·3808	29·665
39	15 21	6·2450	19·748	**89**	79 21	9·4340	29·833
40	16 00	6·3246	20	**90**	81 00	9·4868	30
41	16 81	6·4031	20·248	**91**	82 81	9·5394	30·166
42	17 64	6·4807	20·494	**92**	84 64	9·5917	30·332
43	18 49	6·5574	20·736	**93**	86 49	9·6437	30·496
44	19 36	6·6332	20·976	**94**	88 36	9·6954	30·659
45	20 25	6·7082	21·213	**95**	90 25	9·7468	30·822
46	21 16	6·7823	21·448	**96**	92 16	9·7980	30·984
47	22 09	6·8557	21·679	**97**	94 09	9·8489	31·145
48	23 04	6·9282	21·909	**98**	96 04	9·8995	31·305
49	24 01	7	22·136	**99**	98 01	9·9499	31·464
50	25 00	7·0711	22·361	**100**	100 00	10	31·623

SINES, COSINES, TANGENTS

a	sin *a*	cos *a*	tan *a*	*a*	sin *a*	cos *a*	tan *a*
0	0·0000	1·0000	0·0000	23	0·3907	0·9205	0·4245
0·5	·0087	0·9999	·0087	23·5	·3987	·9171	·4348
1	·0175	·9998	·0175	24	·4067	·9135	·4452
1·5	·0262	·9997	·0262	24·5	·4147	·9100	·4557
2	·0349	·9994	·0349	25	·4226	·9063	·4663
2·5	·0436	·9990	·0437	25·5	·4305	·9026	·4770
3	·0523	·9986	·0524	26	·4384	·8988	·4877
3·5	·0610	·9981	·0612	26·5	·4462	·8949	·4986
4	·0698	·9976	·0699	27	·4540	·8910	·5095
4·5	·0785	·9969	·0787	27·5	·4617	·8870	·5206
5	·0872	·9962	·0875	28	·4695	·8829	·5317
5·5	·0958	·9954	·0963	28·5	·4772	·8788	·5430
6	·1045	·9945	·1051	29	·4848	·8746	·5543
6·5	·1132	·9936	·1139	29·5	·4924	·8704	·5658
7	·1219	·9925	·1228	30	·5000	·8660	·5774
7·5	·1305	·9914	·1317	30·5	·5075	·8616	·5890
8	·1392	·9903	·1405	31	·5150	·8572	·6009
8·5	·1478	·9890	·1495	31·5	·5225	·8526	·6128
9	·1564	·9877	·1584	32	·5299	·8480	·6249
9·5	·1650	·9863	·1673	32·5	·5373	·8434	·6371
10	·1736	·9848	·1763	33	·5446	·8387	·6494
10·5	·1822	·9833	·1853	33·5	·5519	·8339	·6619
11	·1908	·9816	·1944	34	·5592	·8290	·6745
11·5	·1994	·9799	·2035	34·5	·5664	·8241	·6873
12	·2079	·9781	·2126	35	·5736	·8192	·7002
12·5	·2164	·9763	·2217	35·5	·5807	·8141	·7133
13	·2250	·9744	·2309	36	·5878	·8090	·7265
13·5	·2334	·9724	·2401	36·5	·5948	·8039	·7400
14	·2419	·9703	·2493	37	·6018	·7986	·7536
14·5	·2504	·9681	·2586	37·5	·6088	·7934	·7673
15	·2588	·9659	·2679	38	·6157	·7880	·7813
15·5	·2672	·9636	·2773	38·5	·6225	·7826	·7954
16	·2756	·9613	·2867	39	·6293	·7771	·8098
16·5	·2840	·9588	·2962	39·5	·6361	·7716	·8243
17	·2924	·9563	·3057	40	·6428	·7660	·8391
17·5	·3007	·9537	·3153	40·5	·6494	·7604	·8541
18	·3090	·9511	·3249	41	·6561	·7547	·8693
18·5	·3173	·9483	·3346	41·5	·6626	·7490	·8847
19	·3256	·9455	·3443	42	·6691	·7431	·9004
19·5	·3338	·9426	·3541	42·5	·6756	·7373	·9163
20	·3420	·9397	·3640	43	·6820	·7314	·9325
20·5	·3502	·9367	·3739	43·5	·6884	·7254	·9490
21	·3584	·9336	·3839	44	·6947	·7193	·9657
21·5	·3665	·9304	·3939	44·5	·7009	·7133	·9827
22	·3746	·9272	·4040	45	·7071	·7071	1·0000
22·5	·3827	·9239	·4142	45·5	·7133	·7009	1·0176

Values of the functions of angles to 0.1 of a degree (6′) may be found to three places by ordinary interpolation except for tangents of angles near 90°.

SINES, COSINES, TANGENTS

a	sin *a*	cos *a*	tan *a*	*a*	sin *a*	cos *a*	tan *a*
46	0·7193	0·6947	1·0355	**68**	0·9272	0·3746	2·4751
46·5	·7254	·6884	1·0538	**68·5**	·9304	·3665	2·5386
47	·7314	·6820	1·0724	**69**	·9336	·3584	2·6051
47·5	·7373	·6756	1·0913	**69·5**	·9367	·3502	2·6746
48	·7431	·6691	1·1106	**70**	·9397	·3420	2·7475
48·5	·7490	·6626	1·1303	**70·5**	·9426	·3338	2·8239
49	·7547	·6561	1·1504	**71**	·9455	·3256	2·9042
49·5	·7604	·6494	1·1708	**71·5**	·9483	·3173	2·9887
50	·7660	·6428	1·1918	**72**	·9511	·3090	3·0777
50·5	·7716	·6361	1·2131	**72·5**	·9537	·3007	3·1716
51	·7771	·6293	1·2349	**73**	·9563	·2924	3·2709
51·5	·7826	·6225	1·2572	**73·5**	·9588	·2840	3·3759
52	·7880	·6157	1·2799	**74**	·9613	·2756	3·4874
52·5	·7934	·6088	1·3032	**74·5**	·9636	·2672	3·6059
53	·7986	·6018	1·3270	**75**	·9659	·2588	3·7321
53·5	·8039	·5948	1·3514	**75·5**	·9681	·2504	3·8667
54	·8090	·5878	1·3764	**76**	·9703	·2419	4·0108
54·5	·8141	·5807	1·4019	**76·5**	·9724	·2334	4·1653
55	·8192	·5736	1·4281	**77**	·9744	·2250	4·3315
55·5	·8241	·5664	1·4550	**77·5**	·9763	·2164	4·5107
56	·8290	·5592	1·4826	**78**	·9781	·2079	4·7046
56·5	·8339	·5519	1·5108	**78·5**	·9799	·1994	4·9152
57	·8387	·5446	1·5399	**79**	·9816	·1908	5·1446
57·5	·8434	·5373	1·5697	**79·5**	·9833	·1822	5·3955
58	·8480	·5299	1·6003	**80**	·9848	·1736	5·6713
58·5	·8526	·5225	1·6319	**80·5**	·9863	·1650	5·9758
59	·8572	·5150	1·6643	**81**	·9877	·1564	6·3138
59·5	·8616	·5075	1·6977	**81·5**	·9890	·1478	6·6912
60	·8660	·5000	1·7321	**82**	·9903	·1392	7·1154
60·5	·8704	·4924	1·7675	**82·5**	·9914	·1305	7·5958
61	·8746	·4848	1·8040	**83**	·9925	·1219	8·1443
61·5	·8788	·4772	1·8418	**83·5**	·9936	·1132	8·7769
62	·8829	·4695	1·8807	**84**	·9945	·1045	9·514
62·5	·8870	·4617	1·9210	**84·5**	·9954	·0958	10·39
63	·8910	·4540	1·9626	**85**	·9962	·0872	11·43
63·5	·8949	·4462	2·0057	**85·5**	·9969	·0785	12·71
64	·8988	·4384	2·0503	**86**	·9976	·0698	14·30
64·5	·9026	·4305	2·0965	**86·5**	·9981	·0610	16·35
65	·9063	·4226	2·1445	**87**	·9986	·0523	19·08
65·5	·9100	·4147	2·1943	**87·5**	·9990	·0436	22·90
66	·9135	·4067	2·2460	**88**	·9994	·0349	28·64
66·5	·9171	·3987	2·2998	**88·5**	·9997	·0262	38·19
67	·9205	·3907	2·3559	**89**	·9998	·0175	57·29
67·5	·9239	·3827	2·4142	**89·5**	·9999	·0087	114·6
				90	1·0000	·0000	—

ANSWERS TO EXERCISES

The answers to drawing questions are generally given to a slightly higher degree of accuracy than can be expected from the pupil: the last significant figure cannot be expected to be found accurately by drawing.

In calculation exercises 4-figure tables have been used, but results have generally been given to 3 significant figures.

CHAPTER I

Quiz, §1

PAGE

1. **1, 2, 8, 11** are true. The rest are false.

Exercises, §3

5. **2.** The assumption preceding Theorem 5.
3. Yes.
4. (a) (b) (d) are correct; (c) (e) (f) (g) are incorrect. There are three possibilities in (f) and (g) and many possibilities in (c) and (e).

Exercises, §4

8. **7.** $\angle BAC = \angle DAE = 30°$. $\angle B = \angle DAG = 80°$. $\angle C = \angle CAG = 70°$.
8. $\angle DAB = \angle EAF = 33°$. $\angle EAC = \angle DAG = 52°$. $\angle BAC = \angle FAG = 95°$.
9. 89°. **10.** $p - q$.

9. **11.** $\angle BCF = 120°$, other $\angle s = 60°$.
13. $\angle A = \angle C = \angle CBD = 70°$. $\angle ABD = 40°$.
14. $\angle DAC = \angle ACB = 50°$. $\angle DCA = 30°$. $\angle D = \angle ABC = 100°$.
15. $\angle ADC = 78°$. $\angle C = \angle CDE = 72°$. $\angle CAD = \angle ABD = 30°$.

Exercises, §5

10. **1.** (b) line; parallel;
(c) two; two non-parallel;
(d) intersecting; parallel; three; line.

11. (e) parallel; skew (definition);
(f) parallel to; contains;
(g) parallel straight lines.

CHAPTER II

Exercises, §2

PAGE

16. **1.** 7.96 ft. **2.** 5.25 ft.
3. 13.7 mi., 45.5° E. of N. **4.** 28.9 mi., N. 7.7° E.
5. (i) 113.4 ft. (ii) 105.6 ft.

17. **6.** 9.49 ft. **7.** 1.40 ft.
8. 31°, 8.75 ft. **9.** 9 in., 21.1 in., 12.1 in.
10. 23.0 in. **11.** S. 64°45′ E.
12. 31.8°.

18. **13.** 3.23 mi./hr. **14.** 1649 sq. ft.
15. 5.06 ft. **16.** 236 ft.; 18.5°.

Exercises, §3

19. **1.** 169 sq. cm., 13 cm.
2. 25 in.; 5.206 in.; 105.8 mm.; $\sqrt{a^2+b^2}$ mm.
3. 8 cm., 24 sq. cm.; 4.472 in., 8.944 sq. in.;
123.9 mm., 4429 sq. mm.; $\sqrt{x^2-y^2}$ in., $\frac{1}{2}y\sqrt{x^2-y^2}$ sq. in.
4. 36.06 ft.; $\sqrt{a^2+b^2}$ cm.
5. 1.5 in., 3 sq. in.; $\sqrt{x^2-y^2}$ in., $y\sqrt{x^2-y^2}$ sq. in.
6. 21.21 mm.; $x\sqrt{2}$ in., $\sqrt{2}:1$.
7. 7.07 in., 50 sq. in.; $x/\sqrt{2}$ ft., $\frac{1}{2}x^2$ ft.
9. 12.81 mi.; 12.81 mi.; 53·60 km.; 14.14 mi.; 8.60 mi.

20. **10.** 14.38 ft.
11. 10.91 mi.
12. PC = 7.5 mi., AB = 5.292 mi.
13. 1:29.98.
14. AD = 16.97 ft., CE = 8.484 ft.
15. 7.071 mi.
16. 1.414 dm., 1.732 dm.

21. **17.** 17.21 in., 20.58 in., 22.80 in., 24.90 in.
18. 7.383 ft.; 37.21 sq. ft., 55.25 sq. ft.

CHAPTER III

Quiz, §1

22-23. **1, 4, 5, 7, 10, 12, 13, 14** are true. The rest are false.

Exercises, §2

PAGE
24. **1.** (i) CA = 1.833 in., ∠A = ∠C = 73.25°;
(ii) AB = 2.542 in., ∠A = 57.3°, ∠B = 59.2°;
(iii) BC = 5.201 in., ∠B = 21.3°, ∠C = 21.7°;
(iv) BC = 3.030 in., CA = 2.969 in., ∠C = 60°;
(v) CA = 2.001 in., AB = 1.850 in., ∠A = 132°;
(vi) BC = 8.672 cm., BA = 6.261 cm., ∠B = 89.5°;
(vii) ∠A = 44.1°, ∠B = 79.8°, ∠C = 56.1°;
(viii) ∠A = 55.8°, ∠B = 89.4°, ∠C = 34.8°;
(ix) impossible;
(x) impossible;
(xi) AB = 2.720 in., ∠A = 69.3°, ∠C = 77.2°;
or AB = 1.632 in., ∠A = 110.7°, ∠C = 25.8°;
(xii) AB = 2.710 in., ∠A = 47.6°, ∠C = 67.9°.
3. Yes. **4.** No. **5.** 147 yd.
6. 794 yd. **7.** N. 85.5° E., 86.02 mi.

25. **8.** 1232 yd. **9.** 1021 yd.
10. 781.6 ft. **11.** 8.535 mi., 10.96 mi.
12. 75 mi. **13.** 21.97 mi.
14. 3.344 mi., 2.238 mi.

26. **15.** AC = 153 yd., BC = 129 yd.

9.1 10.4

Exercises, §3

27. **1.** N. 79° E., S. 33° E., S. 86° W.
2. S. 37° W., S. 8° E., S. 50° E.
3. N. 42° E.
4. (a) 31 mi., 24 mi.
(b) 50 mi., 35 mi. (roughly).
5. (a) (25.0 mi., N. 69° W.), (13.6 mi., S. 45° W.);
(b) 23.1 mi.;
(c) 13.6 mi. No appreciable difference.

28. **6.** N. 29° E., N. 77° W., N. 68° E., N. 50° W.
7. S. 29° W., N. 28° E.
8. (a) N. 45° E., N. 28° W.;
(b) S. 64° E., S. 56° W.

29. **9.** (Examples.) South out of Cranbrook, 159°M. or 182°T. Northwest out of Lethbridge, 308°M. or 330°T.
11. 39 mi.; 51°M., 74°T.; 231°M.
12. 45 mi.; 300°M., 323°T.; 120°M.
14. L to M (28 mi., 261°), M to P (29 mi., 214°), P to W (30 mi., 154°), W to L (64 mi., 16°).

PAGE

30. **15.** (a) greater; (b) greater.
16. $X = 40^\circ$, $Y = 120^\circ$, $Z = 20^\circ$.
18. (a) (49°37′ N., 114°05′ W.); (b) (49°02′ N., 115°07′ W.); (c) (49°33′ N., 114°38′ W.); (d) (49°26′ N., 112°53′ W.).
19. (49°48′ N., 113°15′ W.).
20. (49°21′ N., 114°10′ W.), (49°26′ N., 113°45′ W.).
21. (a) 196°M., (b) 49°51′ N.

31. **22.** 88°M., 208°M.

CHAPTER IV

Exercises, §1

34. **1.** (i) $\angle BAC$, $\angle BCA$ (fig. 3);
(ii) $\angle CBA$, $\angle CAB$ (fig. 3).

35. **2.** $\angle ACB = 60^\circ$, $\angle ABC = 80^\circ$, $\angle ABD = 100^\circ$.
4. 18° or 102° or 78°.
11. Base ∠s of either Δ are each 70°; exterior angles shown are 70° or 110°.

37. **21.** Congruent in all cases.

CHAPTER V

Exercises, §2

52. **21.** 1.5 in. **26.** 6.40 cm. **29.** 13 ft.
31. regular 8-gon. **32.** regular 6-gon.

CHAPTER VI

Quiz, §1

54. **2, 4, 6** (note new meaning for adjacent angles), **7, 9, 10, 11, 12, 14, 16** are true. The rest are false.

Exercises, §2

57. **2.** (a) 5 parts; (c) 5 parts.
3. (a) 4 rt. ∠s; (b) 4 rt. ∠s; (c) difficulty with meaning of exterior angle.
4. (a) 6 rt. ∠s, 8 rt. ∠s, 2n-4 rt. ∠s. (b) Yes.
6. 82.1°, 97.9°.

PAGE

58. **7.** 4 in. **8.** 1.42 in., 3.24 in.
9. 6.40 cm. **10.** 1.465 in.
11. 50.1°. **12.** 3.99 in.
13. 1.555 hr., 9.79 knots. **14.** N. 33.1° E.

Exercises, §3

60. **1.** 14.8 in.^{2}, 3 in.^{2}, 48.4 cm.^{2}. **2.** 16.66 cm.

61. **6.** 5.880 in.^{2}, 23.53 cm.^{2}, 6 in.^{2}, 2.390 in.^{2}, 5.196 in.^{2}, 6.582 cm.^{2}.
7. 82.5 in.^{2} **8.** 27.72 in.^{2}, 13.16 in.^{2}, 24.46 in.^{2}.
9. 420 in.^{2}. **10.** 7.81 in., 13.42 in., 49.2 in.^{2}.
11. $\frac{1}{2}h(b_1+b_2)$. **12.** 7.894 in.^{2}.
13. 1677 ft.^{2}. **14.** 237 in.^{2}.

62. **15.** 47.5 sq. units. **16.** 3.5 sq. units.
17. 138,300 sq. yd.

63. **18.** 35,000 sq. yd.; 84,400 sq. yd.
19. 2.477 in.^{2}, 14.84 cm.^{2}.
21. 19.29 cm.^{2}. **22.** 296.3 cm.^{2}.

Exercises, §4

64. **1.** yes, no, yes, yes, yes, no. **3.** yes.

65. **4.** 5.831 in., 37.54 mm., $\sqrt{r^2+h^2}$ ft.
5. 19.08 in., 3.666 mm., $\sqrt{l^2-r^2}$ ft.
6. 9.758 cm., $\sqrt{l^2-h^2}$ in. **7.** 5.745 in.
8. 141.4 ft. **9.** 8 in., 13 in., 10.3 in.
10. 8.246 in., 12.04 in., 9.055 in. **11.** 29.2 ft.
12. 16.9 ft.

CHAPTER VII

Exercises, §1

69. **4.** (i) Two parallel planes. (ii) A cylinder.
(iii) A sphere.
(iv) Two spheres. (v) A plane.
(vi) Two planes bisecting the angles between the planes.
(vii) A plane. (viii) Two planes.

70. **8.** A circle. In space, a sphere.

PAGE

Exercises, §2

71. **1.** (a) $0<x<4.5$; $0<y<9$. (b) $2x+y=9$.
2. (a) $y=300+25x$. (b) $0\leqq x\leqq 28$; $300\leqq y\leqq 1000$.

Exercises, §3

72. **1.** 5.66 in. **4.** 9.12 cm.

73. **9.** 68.4 yd. **10.** 352 yd.
11. 1.732 mi. **12.** 34.8°.
13. Yes, with nine possibilities. **16.** 3464 ft.

CHAPTER VIII

Exercises, §1

79. **3.** 1.875 in., 3:5. **4.** $\frac{3}{7}$.
6. $2\frac{1}{7}$, $2\frac{6}{7}$. **7.** $\dfrac{al}{a+b}$, $\dfrac{bl}{a+b}$.
8. 0.3 cm. **9.** 3 in., 4 in.
10. 18 cm. **11.** 6 ft., 9 ft., 12 ft.

80. **14.** 12, $10\frac{5}{6}$, 54. **15.** yes, yes, no, no, yes.

Exercises, §5

88. **1.** 6.4 cm., 8 cm. **2.** 9.2 in., 3.54 in.
3. 24 ft. **4.** 82.5 in.
5. 12.8 ft. **6.** 95.5 ft.

89. **7.** Just visible. **8.** Slope uniform.
9. Embankment, $14\frac{2}{7}$ ft. **10.** (b) 20, $y=\dfrac{100}{x}$.

90. **11.** (b) $\dfrac{10}{3}$, $y=\dfrac{10x}{x+10}$. **12.** (b) $w=\dfrac{5}{9}z$.
(c) $y=\dfrac{5}{4}x$. (d) $y=\sqrt{w^2+25}$. (e) $x=\dfrac{4}{9}t$.
13. $13\frac{5}{7}$ cm.

91. **14.** 2.8 in. **15.** 162.9 sq. in.
16. (i) 18, 24; (ii) 3.75, 4.5. **17.** 5 in., $8\frac{1}{3}$ in.
18. 27 mm., 36 mm.

PAGE

92. **19.** yes, yes, yes, no. **20.** 5.6 cm.
21. UW $=5.05$ in., UV $=5.44$ in.
22. 0.90 mm., 1.71 mm.

93. **23.** 2.625 in., 7.11 in. **24.** $\frac{3}{8}$ in.
25. $5\frac{5}{9}$ in. **26.** (i) $z=5.20$, $y=3.46$.
(ii) $z=\sqrt{3}\,x$. (iii) $y=\sqrt{3}\,(5-x)$.
(iv) $A=2\sqrt{3}x(5-x)$. (v) $0\leqq x\leqq 5$, $0\leqq y\leqq 5\sqrt{3}$,
$0\leqq z\leqq 5\sqrt{3}$, $0\leqq A\leqq 13.5\sqrt{3}$.
27. (c) 10 ft., 19.44 ft.

Exercises, §6

95. **3.** (a) yes; (b) yes; (c) there is only one position of the middle line which makes the two adjacent trapeziums similar; yes.
4. (a) yes; (b) yes; (c) yes.
5. 51 ft.
6. 15.63, 13.75; $y=25-0.375x$; from 0 to 40.

Exercises, §7

96. **1.** (a) (i) nec., (ii) suff. (b) suff.
(c) (i) nec., (ii) suff. (d) nec.
(e) (i) nec., (ii) suff.

97. **2.** No; no. **3.** (i) nec. but not suff.
(ii) suff. but not nec.
4. A, B, C, D suff. for Q; Q is not nec. to A, B, D; C is apparently nec. to Q.

98. **6.** (a) nec. but not suff. (b) as in (a).
(c) both nec. and suff. (d) as in (c).

CHAPTER IX

Exercises, §1

101. **1.** $x=2$, $y=1.155$; $x=3$, $y=1.732$; $r=2y$.
2. (a) $y=x$, $r=1.414x$; (b) $y=1.732x$, $r=2x$.
3. (a) $x^2+y^2=r^2$. (b) r. Except when $a=90°$.
(c) Six. (d) 0; 1. The same.

PAGE

Exercises, §3

106. **3.** 8.66 cm., 7.07 cm., 5 cm.
4. 4.33 ft. **5.** 10.4 ft.
6. 32.4 sq. cm. **7.** $r\sqrt{3}$, 0.5*r*.

Exercises, §5

110. **1.** 7.77; 19.6; 28.0°; $a=7.04$, $b=7.52$; $B=27.3°$, $c=935$.
2-7. See chapter II, §2, ex. 5-10.

111. **8.** 350 ft. **9.** 135 ft., 144 ft.
10. 5500 ft., 595 ft. **11.** 113 ft., 129.8 ft.

Exercises, §6

112. **1.** 0.0947, 5.5°. **2.** Nearly 1°, 0.016, 1.6 per cent.

113. **3.** (*a*) 4.5°; (b) the same.
4. For slopes of 20 per cent. to 100 per cent. the tangent function should be used.

CHAPTER XI

Exercises, §2

147. **2.** 15 in. **3.** 88 in., 132 mi.
4. 24,900 mi. **5.** 22 in.
7. $c/2\pi$. **8.** 7 ft., 9.55 cm.
9. 7 in. **10.** $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{3}$, $\frac{1}{360}$, $\frac{7}{72}$, $\frac{5}{6}$.
11. 4.19 in., 31.43 in. **12.** 0.375, 135°.
13. 3470 mi.

148. **14.** 69.45 mi. **15.** 72°.
16. 5.5 in. **17.** $\frac{11}{18}$ ft.
18. $3\frac{1}{3}$. **19.** $327\frac{3}{11}°$.
21. 12.6 sq. in., 38.5 sq. ft., 707 sq. cm.
22. 4 times. **23.** $\pi:4=0.785:1$.
24. $\pi(R^2-r^2)$. **25.** 163 sq. in.

149. **26.** 3.74 in., 2.13 cm. **27.** 248 yd.
28. 0.75.

Exercises, §3

149. **2**, second part, and **9** no. The rest yes.

PAGE

Exercises, §4

151. **1.** 2.16, 2.5, 3.38 in.
2. Radii 1.90, 5, 2.5 in. In (b) centre is (0, −4).
3. 10.25 ft. **5.** 17.75 ft.

Exercises, §5

151. **1.** 4 cm., $\sqrt{25-a^2}$ cm.

152. **2.** 4 in., $2\sqrt{6.25-d^2}$ in. **3.** 2.5 in.
4. 6, 3.16, 9.49 cm. **5.** 36.25 ft.
6. 8.51, 1.88 cm. **7.** 8.31, 5.77 cm.
8. 2.5, 3.21, 4.70, 5.
10. $r=5$; 2.62, 5.24, 6.98; 6.54, 13.09, 17.45.

153. **12.** 99,392 cu. ft.

Exercises, §6

154. **1.** 7.464 in. **2.** 1500 mi.
3. 3.464 cm. **4.** 5.236 in.
5. 3.464 cm.

Exercises, §7

155. **1.** 1.531, 135°. **2.** 4.24, 5.20 in.; $r\sqrt{2}$, $r\sqrt{3}$.

Exercises, §8

157. **1.** 573.5 ft. **2.** 1146, 383.1, 287.9 ft.
3. 1437, 717, 359 ft. **4.** 2250, 1125, 562.5 ft.; $r\sqrt{2}$ ft.

CHAPTER XII

Exercises, §2

167. **1.** 2.646 cm., 3.464 in., $\sqrt{d^2-r^2}$.
2. $\sqrt{l^2+r^2}$. **3.** circle, radius 5 cm.
4. 3.464 in. **5.** 1.323 in.
6. 15.55 in.

168. **7.** 12 cm. **8.** 10.95 cm.
9. 51.2 ft. **11.** 200 mi.

169. **12.** 15 cm.; yes. **13.** 9.434 cm.
14. 91.2°. **15.** 2.9°.
16. 22.6°, 32.6°. **17.** Angle bisector; 1.1034r.

PAGE

Exercises, §3

171. **1.** 1 in., 2.20 cm. **2.** 52.54 yd.
3. 3.15 in.

172. **10.** 2.29 cm. **11.** 3 in.

Exercises, §4

174. **1.** 29 in., 1 in. **3.** 5, 4, 3 in.
4. 2, 3, 4, 1.453 in.

175. **5.** 11.37 cm. **6.** 8 positions.
8. 3.75 ft. **9.** 4.83 ft.

176. **12.** 1650, 615 yd.

177. **14.** 2/3; 8.33 in.

CHAPTER XIII

Exercises, §2

183. **1.** (i) 112°, 56°; (ii) 55°, 125°, 250°; (iii) 140°, 280°; (v) 152°, 74°; (vi) 45°, 90°, 270°.
3. 20°, 33°, 54°, 73°, 74°, 106°.
4. 30°, 40°, 50°, 60°.

184. **5.** 27°, 35°, 58°, 60°. **6.** 35°, 36°, 73°.

Exercises, §3

185. **1.** . . ., 60°, 40°, 80°. **2.** 90°, 40°, 50°.
3. 60°. **4.** 72°.
6. 110°. **7.** 70°, $(180-2x)^\circ$ or $(2x-180)^\circ$.

Exercises, §5

188. **1.** 1.86 in. **2.** 3.85 cm.
4. 4.06 mi.

189. **5.** 134°.
6. 75°, 15°; 13.9°, 46.1°; impossible; 14.4°, 111.6°.

Exercises, §7

192. **1.** 140°, 70°, 110°. **2.** 64°.

PAGE
193. **3.** 40°, 20°, 50°, 70°.
4. 32°, 36°, 76°, 112°; 41°, 49°, 90°; 73°, 85°, 95°, 107°; 66°, 86°, 94°, 114°.
9. 83°, 63°; $2b+X+Y=180$.

CHAPTER XIV

Exercises, §3

202. **3.** A circle, an ellipse.
4. 31.3 sq. in., 8 cu. in.
5. 9.87 sq. ft., 1.30 cu. ft.; 3.16 ft.

203. **6.** 51.8 sq. ft., 28.3 cu. ft.; 6.18 ft.

Exercises, §4

204. **1.** 6.96 cm., 59.5°, 67.4°. **3.** 20.8 sq. cm.

205. **4.** 11.1°, 5.89 ft.

Exercises, §5

206. **3.** 13 in., 55°. **4.** 2.60 in., 60°.
5. 566 sq. ft. **6.** r/l.
7. 9.66, 16.6, 12.4 cm. **8.** 8 in.

207. **9.** $\frac{R+r}{R-r}\left\{(R-r)^2+h^2\right\}$.

Exercises, §6

208. **2.** 5.33, 0.67 cu. ft.
3. 8 cu. in., 11.66 sq. in., 2.06 in.
5. 34.67 cu. ft.
6. 490 cu. in.

209. **7.** 665 cu. ft. **8.** 27,840 cu. ft.

Exercises, §7

210. **1.** 3,460, 2,490, 1,370 mi.
2. 69.8 mi.; 43.4 mi.; 1 naut. mi. = 1.16 land mi.

211. **3.** 174 mi. **4.** 0.65:1.

PAGE

Exercises, §8

212. **2.** 36.9° N. and S., 11.5° N. and S.
4. $2\pi Rh$, r^2+h^2.
5. 301.6 cu. in.; 2.4 in.; 1:4.
6. 2,830 mi.

Exercises, §9

215. **1.** $\sqrt{2}$ × length of edge.
3. A regular octahedron.

Exercises, §11

219. **3.** 250 sq. ft.
5. 33.75 sq. cm., 33.75 sq. in.
6. 5.6 cm.
7. 144:1.
8. 20 sq. mi.
9. 96 sq. mi.
10. 3.072 acres.
11. 10^4:1.

220. **12.** 87.5 sq. ft.
13. 101, 250 sq. ft.
14. 2,700.
15. 0.8 ft.; 1:2500.

Exercises, §12

221. **4.** 5.6 in.
5. 1.26.
6. 7 min.
7. 7 in.
8. 2.7 × 10^7 gal.
9. 1:1.127.
10. The larger room.
11. 1:1.41.
12. 1:1.44.

Revision Exercises

A

224. **1.** 52.5 ft.
2. 8.6°.
3. 21°.
4. 3.54 in.
5. 9 ft. 5 in.
6. 273, 207, 183 yd.
7. 18,700 yd., 91.4°.

225. **8.** 88°, 38.5°, 53.5°; 260, 162 ft.
9. 2,690 or 2,700 ft.
10. 222 ft.
11. Just possible.
12. 2.09 in.
13. 125 ft. per min.
14. 40.7 ft.

226. **15.** 292 ft.
16. 240 ft. per sec.
17. 11,540 ft.

PAGE

C

229. **44.** 7 in.; 9 in. **45.** 20.6 in.; 18.9 in.
46. 12 in.

230. **47.** 4, 2, 2.67 ft. **48.** 5.33, 0.67, 0.67 in.
49. 15 in. **50.** 7.2 cm.
51. 5.059 in. **52.** 1.8, 3.4 cm.
53. 11.25 cm. **54.** 3.6 ft.
55. 336 lb.

231. **56.** 7.07 ft., 6.36 in.

D

70. 70°. **71.** 26°.
72. $\angle EDB = 15°$, $\angle BEF = 55°$, $\angle DBF = 35°$, $\angle EBD = 75°$.
73. 10.4 cm., 81.1°, outside. **74.** 102.9°.
75. 13.6 sq. in.

233. **76.** . . . Answers are hardly necessary here.
77. (i) 2.24 ft; (ii) $r = 60.8$ ft. **78.** 77.70, 85.14, 85.80 in.
79. 86.0 in. **80.** 38.9°, 2.83 in.
81. 5.14 ft. **82.** 151 mi.; 21°.

234. **83.** 4.61 in.

SUPPLEMENTARY LIST OF DEFINITIONS

Angles of depression and elevation. B is below the level of A. The angle in the vertical plane through A and B from the horizontal line AC through A to the line AB is called the angle of depression of B with respect to A (p. 17).

Since A is above the level of B, $\angle$DBA is the angle of elevation of A with respect to B, where DB is parallel to CA.

Bench mark. In surveying, such as in the layout of water mains, it is necessary to know the height or elevation of the ground above sea level. The survey begins from a bench mark which is a position of known elevation in the given district (p. 18).

Chord. The straight line segment AB joining any two points A, B of a curve is a chord of the curve (see *secant* below).

Contour. A curve of constant height or elevation is said to be a contour line or contour (p. 20).

Imagine a hill and a level path made along the hillside at a height of 1,000 ft. above the valley. This path is a contour line on the hill.

Equiangular figures. (i) A figure is said to be equiangular when all its angles are equal; (ii) two figures are said to be equiangular when corresponding angles are equal (see chap. VIII).

Equivalent figures. When figures have the same area they are said to be equivalent.

Generator of a surface. A straight line which moves under certain conditions to generate a surface is called a generator of the surface. The surface is thus the locus of a straight line. (p, 201) For instance, a line which passes through a fixed point and through the circumference of a circle generates a (double) cone, provided that the point is not in the plane of the circle (p. 53).

The word *generator* may also refer to a straight line segment, such as the generator of a right circular cone (p. 205).

Half-line (*ray*). A line which issues from a point is called a half-line or ray (p. 100).

Median. The line joining the vertex of a triangle to the midpoint of the opposite side of the triangle is called a median. There are three medians.

Rhombus. A parallelogram which has two adjacent sides equal is a rhombus. Are all the sides equal? Is a quadrilateral with four equal sides a rhombus?

Secant. A straight line, other than a tangent, which intersects a curve is called a secant of the curve. The secant is thought of as the entire line.

Trapezium. A quadrilateral which has only one pair of parallel sides is called a trapezium.

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